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# An Optimal Recursive Iterative Algorithm for Discrete Nonlinear Least-Squares Estimation

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# AN OPTIMAL RECURSIVE ITERATIVE ALGORITHM FOR DISCRETE NONLINEAR LEAST-SQUARES ESTIMATION

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The estimation algorithm developed in this paper offers an alternative to standard recursive nonlinear estimators such as the extended Kalman filter and the iterated extended Kalman filter. The algorithm, which is developed from a quadratic cost function basis, splits the problem of cost function minimization into a linear first step and a nonlinear second step by defining new first-step states that are nonlinear combinations of the unknown states. Estimates of the first-step states are obtained by minimizing the first-step cost function using a Kalman filter formulation. Estimates of the unknown, or second-step, states are obtained by minimizing the second-step cost function using an iterative Gauss-Newton algorithm. The two-step estimator is shown to be optimal for static problems in which the time variation of the measurement equation can be separated from the unknowns. This method is then generalized by approximating the nonlinearity as a perturbation of the dynamic update, while keeping the measurement cost function the same. In contrast, the extended Kalman filter and the iterated extended Kalman filter are shown to linearize the measurement cost function, resulting in suboptimal estimates. Two example applications confirm these analytical results.

## Introduction

Most nonlinear least-squares estimation algorithms require a choice between an optimal solution and a recursive formulation. If all the data is available, various nonlinear search and iterative batch algorithms can be used to minimize the desired cost function, resulting in an optimal estimate.[1-6] However, many applications require a recursive formulation of the estimation problem. The standard nonlinear recursive estimators, such as the extended Kalman filter (EKF) and the iterated extended Kalman filter (IEKF),[7] linearize the cost function in order to use the well-known, linear Kalman filter[8] equations. This linearization, however, results in an estimate that is biased—*i.e.*, the expected value of the estimator is not the true value of the states—and suboptimal. On the other hand, the two-step estimator developed in this paper obtains the optimal estimate with a recursive algorithm by breaking the cost function minimization into two steps and moving the nonlinearity to the second step.

In contrast to the nonlinear least-squares problem, methods for finding the optimal estimate for the linear problem are well established. For the static case, in which the unknown parameters are constant, the linear weighted least-squares batch fit provides the optimal estimate. When there are dynamics associated with the unknown parameters, the Kalman filter gives the optimal estimate at the current time step. The optimal estimate for previous time steps based on all the data can be found using a smoothing algorithm,[9] which combines past, present and future measurements in off-line (post-processing) calculations. If the measurement noise is Gaussian, the Kalman filter is equivalent to both the minimum-variance and maximum-likelihood solutions and results in an unbiased estimate.[9]

When the unknown parameters enter the measurement equation in a nonlinear manner, the problem of estimation becomes more difficult. (Problems with nonlinear dynamics are not addressed in this paper. See Refs. 7 and 10 for approaches to these problems.) If the estimation can be done off-line, there are several optimal methods available. For the static case, an iterative batch technique or nonlinear search algorithm can be used.[1-2] For the dynamic problem, an iterated linearized smoother can be employed to obtain the optimal estimate.[3-6] The advantage of off-line estimation methods is that all of the data can be used to obtain the optimal estimates of the states. There are, however, several disadvantages associated with these methods. First, many applications require real-time state estimates that, for example, can be used in dynamic feedback loops in control problems. Second, it is difficult to update an estimate with new data without using the entire data set again. Finally, the dependencies of the state estimates on other parameters are obscured. There is

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often much insight that can be gained by observing the time variation of the estimates in recursive algorithms.

As a result, a number of approximate recursive algorithms have been developed.[7,11] The two most common, the EKF and the IEKF, are examples of methods which make approximations to the measurement equation in order to allow use of the familiar linear filter equations. These methods give on-line state estimates, thus facilitating their use in real-time applications and the update of current estimates with new data. Unfortunately, these methods result in biased, suboptimal state estimates because they linearize the original quadratic cost function in order to develop a recursive algorithm. This linearization, by assuming the errors to be small, makes these algorithms very sensitive to the *a priori* estimate of the unknown states about which the cost function is linearized.

The two-step estimator developed in this paper provides an alternative to the preceding algorithms. It avoids linearizing the cost function by breaking the minimization into two steps. A new set of states is defined for the first step using nonlinear combinations of the unknown, or second-step, states, so that the measurement equation becomes linear in the first-step states. The linear first-step problem can be solved optimally and recursively using a Kalman filter. The second-step states are then optimally obtained by treating the first-step state estimates as measurements and using an iterative Gauss-Newton algorithm to perform a nonlinear least-squares fit. For static problems in which the time variation of the measurement equation can be separated from the unknown parameters, no approximation is necessary and the two-step algorithm achieves the optimal estimate. For other static problems or problems with linear dynamics, an approximation of the measurement nonlinearity is made in the time update, but the measurement cost function remains unchanged. The result for the problems examined here is a substantial reduction in estimate bias and sensitivity to initial conditions when compared to the EKF and the IEKF.

This paper presents a summary of the two-step estimator, including comparisons with some of the other methods described above. First, the nonlinear least-squares estimation problem is described. Next, the two-step solution to the static problem with separable time variation is presented. Included are a summary of the method's cost function basis, a proof of optimality, and derivations of the equations used to perform the optimization of the first and second steps. The method is then enhanced to include other static problems and problems with linear state dynamics by deriving an approximate time update for the first-step states. The problem of how to choose the first-step states is addressed and a general method is presented. Implementation of the two-step estimator is summarized. An analytical comparison is made in which the EKF and the IEKF are shown to be biased and suboptimal. Finally, we give two example applications, one a static parameter estimation problem from the Stanford Gravity Probe B Relativity Mission (GP-B) and the other a dynamic problem of radar ranging. Numerical comparisons of the different methods are made using these applications.

## Nonlinear Least-Squares Estimation

The general problem of parameter estimation from discrete nonlinear measurements can be described by state-space dynamic and measurement equations of the form

$$x_{k+1} = \Phi_k x_k + \Gamma_k w_k \tag{1}$$

$$z_k = F(x_k, t_k) + v_k \tag{2}$$

where the subscript $k$ denotes the value at time step $t_k$. The dynamic model in Eq. (1) describes how the unknown state vector $x$ is propagated through time by the state transition matrix $\Phi$ and the process noise forcing function $\Gamma w$. In Eq. (2), the measurements $z$ are expressed as functions $F$ of the unknown states and time, plus measurement noise $v$. The process noise and measurement noise are assumed to be white, Gaussian processes with zero mean and covariance matrices $Q$ and $R$, respectively (*i.e.*, $E[ww^T] = Q$ and $E[vv^T] = R$). There may also be *a priori* information available in the form of an initial guess $\bar{x}_1$ and an initial covariance matrix $M_1$.

The optimality of an estimate of the unknown parameters is determined with respect to a cost function $J$ defined over all the measurements. The optimal estimate is the vector that minimizes this cost function subject to the constraints of Eq. (1). In the least-squares problem, a quadratic cost function is used

$$J = \frac{1}{2}(x_1 - \bar{x}_1)^T M_1^{-1}(x_1 - \bar{x}_1) + \frac{1}{2}\sum_{k=1}^{N-1} w_k^T Q_k w_k + \frac{1}{2}\sum_{k=1}^{N}(z_k - F(x_k, t_k))^T R_k^{-1}(z_k - F(x_k, t_k)) \tag{3}$$

so that $J$ is a weighted sum of squares of the initial guess, the process noise and the residuals $z - F(x,t)$. $N$ is the total number of time steps. Justification of the nonlinear quadratic cost function is given by the fact that the optimal estimate is equivalent to the joint maximum-likelihood solution[11] and can be shown to approach the minimum-variance solution

asymptotically as the number of measurements increases.[12]

## The Optimal Static Solution

Minimization of the cost function in Eq. (3) subject to the constraints of Eq. (1) can be done by adjoining these equations to the cost function using Lagrange multipliers.[9] In the linear problem, this leads to the Kalman filter. The nonlinear problem can be solved optimally in a batch implementation by an iterated smoother.[5] The traditional recursive approaches, such as the EKF and the IEKF, attempt to minimize a quadratic cost function at each time step based on the estimate from the previous time step but ultimately fail to minimize the cost function in Eq. (3). The goal of this work was to obtain a recursive algorithm that would more closely approach the optimal batch estimate.

To simplify the problem temporarily, we will first address the static problem, in which $\Phi$ is the identity matrix and $Q = 0$. Under the additional assumption that the time variation of the measurement equation can be separated from the unknown states, we show that the two-step estimator offers an optimal, recursive algorithm. The GP-B data reduction problem, which was the inspiration for the estimation algorithm presented here, is an example of such a problem and is described in greater detail in the first example. This optimal static solution forms the basis of the two-step method. It is later augmented in an approximate manner to address problems in which the time variation and unknowns cannot be separated, including both the more general static problem as well as problems with state dynamics. The estimates obtained by this augmented method will then be compared to the optimal estimate as defined by Eq. (3).

Before rewriting our cost function for the static problem, we will define some new variables to simplify the derivation. They are

$$\xi = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} \quad \mathcal{F}(x,t) = \begin{bmatrix} F(x,t_1) \\ \vdots \\ F(x,t_N) \end{bmatrix} \quad \rho = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} \tag{4}$$

The measurement equation can now be rewritten to include all the time steps

$$\xi = \mathcal{F}(x,t) + \rho \tag{5}$$

The cost function in Eq. (3) becomes

$$J = \frac{1}{2}(\xi - \mathcal{F}(x,t))^T \mathcal{R}^{-1}(\xi - \mathcal{F}(x,t)) \tag{6}$$

where the state dynamics have been eliminated in the static problem. We have also assumed no *a priori* information to simplify the derivation. The weighting matrix is defined by $E[\rho\rho^T] = \mathcal{R}$, so that $\mathcal{R}$ is a block diagonal matrix with the matrices $R_k$ on the diagonal. Equations (5) and (6) define the static nonlinear estimation problem.

A necessary condition for optimality is that the estimate be a stationary point of the cost function,[1] *i.e.*, the derivative of $J$ must vanish at the optimal estimate $\hat{x}$

$$\left.\frac{\partial J}{\partial x}\right|_{x=\hat{x}} = 0 \tag{7}$$

Taking the derivative of Eq. (6) with respect to the state vector $x$ yields

$$\frac{\partial J}{\partial x} = -(\xi - \mathcal{F}(x,t))^T \mathcal{R}^{-1} \frac{\partial \mathcal{F}(x,t)}{\partial x} \tag{8}$$

The optimal estimate of the unknown parameters is then the vector that makes the right-hand side of this equation vanish.

## Two-Step Estimator

The two-step nonlinear least-squares estimator divides the problem of minimizing the cost function into a linear first step, estimating nonlinear combinations of the unknown states using a linear least-squares fit, and a nonlinear second step, employing an iterative, nonlinear least-squares fit of the first-step estimates to obtain estimates of the unknown states.

The first-step states are defined as a vector of nonlinear functions of the unknown states $y = f(x)$. We assume that these functions can be chosen so that the time variation is separated from the unknowns and the measurement equation (Eq. (2)) becomes linear in the first-step states

$$z_k = H_k y + v_k \tag{9}$$

The method for choosing these first-step states is problem dependent and is addressed in a later section. This measurement equation can be expanded as in the previous section to include all the time steps by defining a matrix $\mathcal{H}$ that comprises all the $H_k$ matrices. The result is that the nonlinear function $\mathcal{F}(x,t)$ in Eqs. (5) and (6) is replaced by $\mathcal{H}y$, where $\mathcal{H}$ depends explicitly on time. Eq. (5) becomes

$$\xi = \mathcal{H}y + \rho \tag{10}$$

The cost function for estimating the first-step states becomes

$$J_y = \frac{1}{2}(\xi - \mathcal{H}y)^T \mathcal{R}^{-1}(\xi - \mathcal{H}y) \tag{11}$$

Minimizing this cost function yields the optimal estimate $\hat{y}$ and its associated covariance matrix $P_y$. Note that this first step is completely linear.

The estimates of the first-step states are treated as measurements in the second step. The second-step measurement equation is

$$\hat{y} = f(x) + e \tag{12}$$

where the measurement noise $e$ has covariance matrix $P_y$, the covariance of the first-step states. The second-step cost function is then defined as

$$J_x = \frac{1}{2}(\hat{y} - f(x))^T P_y^{-1}(\hat{y} - f(x)) \tag{13}$$

Minimizing this nonlinear quadratic cost function yields the optimal estimate $\hat{x}$ of the second-step states, as well as a covariance matrix $P_x$ describing the error associated with the estimate. Equations (10)-(13) provide the basis from which the two-step estimator is derived.

Proof of Optimality

Consecutive minimization of the quadratic cost functions $J_y$ and $J_x$ can be shown to be equivalent to minimizing the original cost function in Eq. (6). As before, a necessary condition for optimization is that the derivative of the cost function vanish at the optimal estimate. Since the first step is linear, the optimal estimate is just the well-known linear weighted least-squares solution

$$\hat{y} = P_y \mathcal{H}^T \mathcal{R}^{-1} \xi \tag{14}$$

with covariance matrix given by

$$P_y = (\mathcal{H}^T \mathcal{R}^{-1} \mathcal{H})^{-1} \tag{15}$$

The second-step cost function is minimized by setting its derivative to zero. Taking the derivative of $J_x$ yields

$$\frac{\partial J_x}{\partial x} = -(\hat{y} - f(x))^T P_y^{-1} \frac{\partial f(x)}{\partial x} \tag{16}$$

Substituting from Eqs. (14) and (15), this can be rewritten as

$$\frac{\partial J_x}{\partial x} = -(\xi - \mathcal{H}f(x))^T \mathcal{R}^{-1} \frac{\partial \mathcal{H}f(x)}{\partial x} \tag{17}$$

Finally, substituting $\mathcal{F}(x,t)$ for $\mathcal{H}f(x)$ yields Eq. (8). Thus minimizing the cost functions $J_y$ and $J_x$ in this manner is equivalent to minimizing the original cost function $J$.

First-Step Optimization

Since the first-step cost function is linear, an optimal estimate of the first-step states can be obtained with either a batch fit or, if a recursive formulation is desired, a Kalman filter. The batch fit is just the weighted least-squares solution given by Eqs. (14) and (15). It can be shown that the Kalman filter reduces to the weighted least-squares fit for the static problem,[3] and thus provides an optimal, recursive alternative. The static Kalman filter is given by

Measurement Update:

$$\begin{aligned} \hat{y}_k &= \bar{y}_k + P_{y_k} H_k^T R_k^{-1}(z_k - H_k \bar{y}_k) \quad k = 1 \ldots N \\ P_{y_k} &= (M_{y_k}^{-1} + H_k^T R_k^{-1} H_k)^{-1} \end{aligned} \tag{18}$$

Time Update:

$$\begin{aligned} \bar{y}_{k+1} &= \hat{y}_k \quad k = 1 \ldots N-1 \\ M_{y_{k+1}} &= P_{y_k} \end{aligned}$$

The equivalence of the static Kalman filter and the weighted least-squares fit implies that $\hat{y}_N$ from Eq. (18) is equal to $\hat{y}$ in Eq. (14). It is important to note that any of the alternative forms of the Kalman filter that may be computationally better (*e.g.*, information filters, square root algorithms, etc.)[9,13] can be substituted for the form given in Eq. (18) since this is a linear problem.

Second-Step Optimization

Optimization of the second step can be carried out whenever the best current estimate of the second-step states is desired (it need not be done at every time step of the Kalman filter for the optimal static problem). An iterative algorithm is used to find the optimal estimate in the nonlinear second step. One such method is the Newton-Raphson algorithm,[1] which can be written as

$$\hat{x}_{i+1} = \hat{x}_i - \mathrm{H}_i^{-1} q_i^T \tag{19}$$

where $i$ is the iteration number and $q_i$ is the gradient of the cost function $J_x$ evaluated at $\hat{x}_i$

$$q_i = \left.\frac{\partial J_x}{\partial x}\right|_{x=\hat{x}_i} \tag{20}$$

$\mathrm{H}_i$, the Hessian, is the second derivative of $J_x$, again evaluated at $\hat{x}_i$

$$\mathrm{H}_i = \left.\frac{\partial^2 J_x}{\partial x^2}\right|_{x=\hat{x}_i} \tag{21}$$

The gradient $q$ is given by Eq. (16). Taking another derivative of this to obtain the Hessian would result in second derivatives of the vector of functions $f$. This is difficult to compute and does not necessarily give a positive definite Hessian, which is required for convergence. Positive definiteness can be ensured by ignoring the terms with second derivatives of $f$ and using the Gauss approximation of the Hessian

$$\mathrm{H}_i \approx \mathrm{H}_{G_i} = \left.\frac{\partial f}{\partial x}\right|_{x=\hat{x}_i}^T P_y^{-1} \left.\frac{\partial f}{\partial x}\right|_{x=\hat{x}_i} \tag{22}$$

Using this approximation, Eq. (19) becomes

$$\hat{x}_{i+1} = \hat{x}_i - \mathrm{H}_{G_i}^{-1} q_i^T \tag{23}$$

which can be iterated until $\hat{x}_{i+1} - \hat{x}_i \to 0$, or some other convergence criterion is reached. This algorithm is known as the Gauss-Newton method.

An approximate covariance associated with this estimate can be derived from Eq. (23). The gradient $q_i$ is obtained by evaluating Eq. (16) at $\hat{x}_i$. Substituting this into Eq. (23) yields

$$\hat{x}_{i+1} = \hat{x}_i + \mathrm{H}_{Gi}^{-1} \left.\frac{\partial f}{\partial x}\right|_{x=\hat{x}_i}^{T} P_y^{-1}(\hat{y} - f(\hat{x}_i)) \qquad (24)$$

The vector of functions $f$ can be expanded in a Taylor series about the current estimate. Truncating the series after the second term gives

$$f(x) \approx f(\hat{x}_i) + \left.\frac{\partial f}{\partial x}\right|_{x=\hat{x}_i} (x - \hat{x}_i) \qquad (25)$$

This result is substituted into Eq. (12), which in turn is substituted into Eq. (24). After some algebraic manipulation and use of the Gauss approximation of the Hessian (Eq. (22)), Eq. (24) can be rewritten as

$$\hat{x}_{i+1} \approx x + \mathrm{H}_{Gi}^{-1} \left.\frac{\partial f}{\partial x}\right|_{x=\hat{x}_i}^{T} P_y^{-1} e \qquad (26)$$

where $e = \hat{y} - f(x)$, from Eq. (12). The covariance of the states $x$ can be calculated from the expected value

$$P_x = E[(\hat{x}_{i+1} - x)(\hat{x}_{i+1} - x)^T] \qquad (27)$$

Substituting Eq. (26) into Eq. (27) and using the fact that $E[ee^T] = P_y$, the covariance is given by

$$P_x \approx \mathrm{H}_{Gi}^{-1} \qquad (28)$$

This covariance is approximate because of the truncation of the Taylor series expansion of $f$. The approximation improves as the estimate of $x$ approaches the true value.

Equations (23) and (28) form the iterative second-step optimization. Together with Eq. (18), they constitute the optimal two-step estimator. It is interesting to note that for cases where the number of first-step states is equal to the number of second-step states, it is possible to choose the second-step estimates so that the cost function vanishes. Depending on the type of nonlinearity in the problem, it is thus sometimes possible to invert the functions $f$ to get $\hat{x}$ as a function of $\hat{y}$, making iteration to find the minimum unnecessary.

## Dynamic Solution

In order to generalize the optimal static solution derived above, we will now relax our assumption that the first-step states do not vary with time. This time variation can be either explicit, in that time enters directly into the nonlinear vector of functions $f$, or implicit, through the dynamics of the second-step states $x$. The generalization is accomplished by making an approximation to the nonlinearity, resulting in the time variation of the first-step states being treated like a process noise perturbation in the first-step time update. This more general formulation expands the range of problems to which the two-step estimator can be applied, while retaining the optimal static solution as a special case. Both solutions assume that the second-step states are observable from the first-step estimates. This issue affects the choice of first-step states and will be addressed in the next section.

The static solution derived above is optimal because the two steps are decoupled. This keeps the nonlinearity of the second step from affecting the linear first step, where the noise averaging is being done. For problems with time-varying first-step states, this optimality could be maintained if we could exactly describe the dynamics of the first-step states with a linear formula that was independent of the second-step states. We would then use the optimal dynamic Kalman filter. However, the nonlinearity of the problem generally prevents us from keeping the two steps decoupled, and the first-step dynamics must be approximated. This can be accomplished using the known relationship between the first- and second-step states

$$y_k = f(x_k, t_k) \qquad (29)$$

The explicit time dependence of the first-step states is shown in Eq. (29). In the sequel, the notation $f_k$ will be used to indicate this explicit time dependence. The dynamics of the second-step states are given by Eq. (1). The optimal time update for the second-step states[9] is

$$\begin{aligned} \bar{x}_{k+1} &= \Phi_k \hat{x}_k \qquad k = 1 \ldots N-1 \\ M_{x_{k+1}} &= \Phi_k P_{x_k} \Phi_k^T + \Gamma_k Q_k \Gamma_k^T \end{aligned} \qquad (30)$$

This time update, along with Eq. (29), can be used to obtain an approximate time update for the first-step states. The goal is to obtain a dynamic update equation in the first-step states $y$ similar to Eq. (1), from which the optimal time update can be derived easily. Using the identity in Eq. (29) at time steps $k$ and $k+1$ yields

$$y_{k+1} = y_k + f_{k+1}(x_{k+1}) - f_k(x_k) \qquad (31)$$

The last two terms can be approximated by a first-order Taylor series representation

$$\begin{aligned} f_k(x_k) &\approx f_k(\hat{x}_k) + \left.\frac{\partial f_k}{\partial x_k}\right|_{x_k=\hat{x}_k} (x_k - \hat{x}_k) \\ f_{k+1}(x_{k+1}) &\approx f_{k+1}(\bar{x}_{k+1}) + \left.\frac{\partial f_{k+1}}{\partial x_{k+1}}\right|_{x_{k+1}=\bar{x}_{k+1}} (x_{k+1} - \bar{x}_{k+1}) \end{aligned} \qquad (32)$$

The time update of the states is the found from the expectation

$$\bar{y}_{k+1} = E[y_{k+1}] \tag{33}$$

Substituting Eqs. (31) and (32) into Eq. (33) and using the approximations that $E[x_k] \approx \hat{x}_k$ and $E[x_{k+1}] \approx \bar{x}_{k+1}$ (an equality for the linear solution) yields

$$\bar{y}_{k+1} \approx \hat{y}_k + f_{k+1}(\bar{x}_{k+1}) - f_k(\hat{x}_k) \tag{34}$$

The covariance matrix of this time update is defined as

$$M_{y_{k+1}} = E[(y_{k+1} - \bar{y}_{k+1})(y_{k+1} - \bar{y}_{k+1})^T] \tag{35}$$

Substituting Eqs. (31), (32) and (34) into Eq. (35), and using the approximations that $E[x_k] \approx \hat{x}_k$ and $E[x_{k+1}] \approx \bar{x}_{k+1}$ and the following definitions

$$\begin{aligned} P_{x_k} &= E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] \\ M_{x_{k+1}} &= E[(x_{k+1} - \bar{x}_{k+1})(x_{k+1} - \bar{x}_{k+1})^T] \end{aligned} \tag{36}$$

the covariance update can be written as

$$\begin{aligned} M_{y_{k+1}} \approx P_{y_k} &+ \left.\frac{\partial f_{k+1}}{\partial x_{k+1}}\right|_{x_{k+1}=\bar{x}_{k+1}} M_{x_{k+1}} \left.\frac{\partial f_{k+1}}{\partial x_{k+1}}\right|^T_{x_{k+1}=\bar{x}_{k+1}} \\ &- \left.\frac{\partial f_k}{\partial x_k}\right|_{x_k=\hat{x}_k} P_{x_k} \left.\frac{\partial f_k}{\partial x_k}\right|^T_{x_k=\hat{x}_k} \end{aligned} \tag{37}$$

Equations (34) and (37) constitute the approximate time update for the first-step states. The approximation improves as $\hat{x}_k \to x_k$ (gets better as the number of measurements increases) and $\bar{x}_{k+1} \to x_{k+1}$ (gets better as the process noise gets smaller and the number of measurements increases). Increasing the sampling frequency should also improve the approximation when there is an explicit time dependence in the first-step states. As the approximation gets better, the estimate will approach the optimal value. Comparing the form of Eq. (37) to that of Eq. (30) shows that the time variation of the first-step states has essentially been treated as a perturbation term, entering the time update in the same manner as the process noise.

It is interesting to note a couple of special cases of this time update. If $f$ is not an explicit function of time and there is only process noise in the state dynamics ($\Phi$ is the identity matrix), the covariance update reduces to

$$M_{y_{k+1}} \approx P_{y_k} + \left.\frac{\partial f}{\partial x_k}\right|_{x_k=\hat{x}_k} \Gamma_k Q_k \Gamma_k^T \left.\frac{\partial f}{\partial x_k}\right|^T_{x_k=\hat{x}_k} \tag{38}$$

As expected, the time update of the covariance matrix involves only a correction term due to process noise, resulting in an increase in the covariance estimates of those first-step states affected by process noise. If the process noise is also removed ($Q = 0$), the dynamic time update collapses to the static time update given in Eq. (18), as expected.

## Choosing First-Step States

An implicit assumption of the two-step estimator derived above has been that the second-step states are observable from the first-step states. If this were not the case, the Hessian matrix of Eq. (22) would not be full rank and thus could not be inverted to perform the iterative update. The choice of first-step states is dependent on the particular problem being addressed. We have devised a two-stage sequence that provides a general method for making this choice. The first stage is to attempt to separate the time variation from the unknown parameters through algebraic manipulation. If this can be accomplished, the optimal version of the two-step estimator can be used. The static example given below provides a good example of a problem which is amenable to this method. One general group of problems of this nature is the estimation of amplitudes and phases of sinusoidal signals, which can be derived in a manner similar to that of the static example.

If sufficient separation of the time variation cannot be done, then the second stage is to add some or all of the second-step states to the first-step state vector. The simplest version of this idea is to use the measurement function as one first-step state and then add the entire second-step state vector as the rest of the first-step states

$$y_k = \begin{bmatrix} F(x_k, t_k) \\ x_k \end{bmatrix} \tag{39}$$

Clearly the second-step states are observable now. The measurement matrix $H_k$ in this case is a row vector with a one in the first postion followed by $n$ zeros, where $n$ is the number of unknown states $x$. It is important to note that while we now have estimates of our unknowns after the first-step optimization, these estimates are not optimal. There is still information about these states contained in the estimate of the first state in Eq. (39). This information must be combined optimally with the estimates of the other first-step states, using the associated covariance matrix. This is precisely what the second step of the two-step algorithm does. The dynamic example given below illustrates how Eq. (39) can be used to handle problems with implicit time variation due to state dynamics. A good example of a problem that would have explicit time variation in the first-step states (as opposed to state dynamics) would be estimation of the amplitude, phase and frequency of a sine wave. The measurement equation can be writtten as

$$z_k = A\cos(\omega t_k + \phi) + v_k \tag{40}$$

where the constant unknown state vector

$$x = \begin{bmatrix} A \\ \omega \\ \phi \end{bmatrix} \tag{41}$$

includes the amplitude $A$, the frequency $\omega$ and the phase $\phi$. The measurement equation can be rewritten following the method of Eq. (39) as

$$z_k = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A\cos(\omega t_k + \phi) \\ A \\ \omega \\ \phi \end{bmatrix} + v_k = Hy_k + v_k \tag{42}$$

where the first-step states are now a function of the second-step states, $y_k = f_k(x)$. This problem is not simulated in this paper.

By their nature, nonlinear problems tend to require a problem-specific approach. The method of Eq. (39) should allow implementation of the two-step formulation on any problem that the extended Kalman filter can handle. However, better methods of choosing the states may be possible for particular nonlinearities, making the choice of first-step states an engineering design issue.

## Summary of Two-Step Estimator

The two-step estimator can be summarized as follows:

Given:

Measurements:

$$z_k = F(x_k, t_k) + v_k = H_k y_k + v_k \qquad k = 1 \ldots N$$

Dynamics:

$$x_{k+1} = \Phi_k x_k + \Gamma_k w_k \qquad k = 1 \ldots N-1$$

Nonlinearity: $y_k = f_k(x_k)$

Initial Conditions: $\bar{y}_1, M_{y_1}$

where $E[v_k] = 0 \quad E[v_k v_k^T] = R_k$

$E[w_k] = 0 \quad E[w_k w_k^T] = Q_k$

First - Step Optimization:

Measurement Update:

$$\hat{y}_k = \bar{y}_k + P_{y_k} H_k^T R_k^{-1} (z_k - H_k \bar{y}_k) \qquad k = 1 \ldots N$$

$$P_{y_k} = (M_{y_k}^{-1} + H_k^T R_k^{-1} H_k)^{-1}$$

Time Update:

$$\bar{y}_{k+1} = \hat{y}_k + f_{k+1}(\bar{x}_{k+1}) - f_k(\hat{x}_k) \qquad k = 1 \ldots N-1$$

$$M_{y_{k+1}} = P_{y_k} + \left.\frac{\partial f_{k+1}}{\partial x_{k+1}}\right|_{x_{k+1}=\bar{x}_{k+1}} M_{x_{k+1}} \left.\frac{\partial f_{k+1}}{\partial x_{k+1}}\right|^T_{x_{k+1}=\bar{x}_{k+1}} - \left.\frac{\partial f_k}{\partial x_k}\right|_{x_k=\hat{x}_k} P_{x_k} \left.\frac{\partial f_k}{\partial x_k}\right|^T_{x_k=\hat{x}_k}$$

Second - Step Optimization:

Measurement Update:

$$\hat{x}_{k,i+1} = \hat{x}_{k,i} - \mathrm{H}_{G_{k,i}}^{-1} q_{k,i}^T \qquad k = 1 \ldots N$$

$$\mathrm{H}_{G_{k,i}} = \left.\frac{\partial f_k}{\partial x_k}\right|^T_{x_k=\hat{x}_{k,i}} P_{y_k}^{-1} \left.\frac{\partial f_k}{\partial x_k}\right|_{x_k=\hat{x}_{k,i}}$$

$$q_{k,i} = -(\hat{y}_k - f_k(\hat{x}_{k,i}))^T P_{y_k}^{-1} \left.\frac{\partial f_k}{\partial x_k}\right|_{x_k=\hat{x}_{k,i}}$$

$$P_{x_{k,i}} = \mathrm{H}_{G_{k,i}}^{-1}$$

Time Update:

$$\bar{x}_{k+1} = \Phi_k \hat{x}_k \qquad k = 1 \ldots N-1$$

$$M_{x_{k+1}} = \Phi_k P_{x_k} \Phi_k^T + \Gamma_k Q_k \Gamma_k^T$$

where $i$ represents iteration number in the second-step optimization. It is important to note that the second-step optimization must be carried out between the measurement and time updates of the first-step optimization at each time step for problems in which the time variation is not separable. In the case of separable time variation, the first-step optimization reduces to the static Kalman filter in Eq. (18) and the second-step optimization need not be done at every time step, as the two steps are now decoupled.

## Comparison to Extended Kalman Filter and Iterated Extended Kalman Filter

The motivation for the two-step method was to devise a recursive estimator for nonlinear problems that would provide an optimal alternative to the extended Kalman filter (EKF) or the iterated extended Kalman filter (IEKF) typically applied to such problems. The EKF and the IEKF are biased, suboptimal estimators because they approximate the original cost function by linearizing about the previous estimate, thereby reaching a false minimum. This can be shown by deriving the EKF and the IEKF for the static problem (similar results can be derived for the dynamic case) from the cost function at time step $k$

$$J_k = \frac{1}{2}[(x - \bar{x}_k)^T M_k^{-1} (x - \bar{x}_k) + (z_k - F_k(x))^T R_k^{-1} (z_k - F_k(x))] \tag{43}$$

where the subscript $k$ on the function $F$ indicates explicit dependence on time. For the algorithm to be optimal, it must not only minimize this cost function at each time step, but it must also optimally combine the estimates from each time step to achieve minimization of the original cost function in Eq. (6) defined over all the measurements. As stated previously, if $F$ is a linear function of the states, the Kalman filter, which can be derived from a linear version of Eq. (43),[9] obtains the optimal estimate and is thus equivalent to the weighted least-squares batch fit. The Kalman filter is able to

combine optimally the cost functions at each time step because the state and covariance equations are decoupled. The EKF and the IEKF, on other hand, linearize about the current estimate and thus cannot decouple the state and covariance estimates. The following derivations show that the IEKF obtains the optimal estimate at each time step (*i.e.*, it minimizes the cost function in Eq. (43)), but does not minimize the original cost function, while the EKF does not even obtain the optimal estimate at each time step. Both methods are shown to give biased estimates at each time step. These proofs provide an analytical basis for claims that the two-step estimator is an improvement over the EKF and the IEKF.

The EKF can be derived by expanding the nonlinear function $F_k$ in a Taylor series about the initial guess

$$F_k(x) \approx F_k(\bar{x}_k) + \left.\frac{\partial F_k}{\partial x}\right|_{x=\bar{x}_k} (x - \bar{x}_k) \tag{44}$$

Substituting this into Eq. (43) gives a linear cost function

$$J_k = \frac{1}{2}[(x-\bar{x}_k)^T M_k^{-1}(x-\bar{x}_k) + (z_k' - H_k x)^T R_k^{-1}(z_k' - H_k x)] \tag{45}$$

where

$$z_k' = z_k - F_k(\bar{x}_k) + H_k \bar{x}_k \tag{46}$$

and

$$H_k = \left.\frac{\partial F_k}{\partial x}\right|_{x=\bar{x}_k} \tag{47}$$

Minimizing this approximate cost function by making its derivative vanish yields the EKF measurement update

$$\begin{aligned} \hat{x}_k &= \bar{x}_k + P_k H_k^T R_k^{-1}(z_k - F_k(\bar{x}_k)) \\ P_k &= (M_k^{-1} + H_k^T R_k^{-1} H_k)^{-1} \end{aligned} \tag{48}$$

which is the same as that derived in Ref. 7. This solution is suboptimal because it does not minimize the desired cost function at each time step (Eq. (43)), and thus does not minimize the original cost function in Eq. (6).

The EKF estimator can also be shown to be biased. Equation (44) can be made into an equality by adding an error term $b_E$ to the right-hand side. The new equality can be written as

$$b_E = F_k(x) - F_k(\bar{x}_k) - \left.\frac{\partial F_k}{\partial x}\right|_{x=\bar{x}_k} (x - \bar{x}_k) \tag{49}$$

If there is no *a priori* information, then $M_k^{-1} = 0$. The measurement covariance update becomes

$$P_k = (H_k^T R_k^{-1} H_k)^{-1} \tag{50}$$

Substituting Eqs. (49) and (50) into the state measurement update of Eq. (48) and using the fact that $z_k = F_k(x) + v_k$, the state measurement update can be written as

$$\hat{x}_k = x + P_k H_k^T R_k^{-1}(v_k + b_E) \tag{51}$$

where the covariance matrix $P_k$ and the gradient matrix $H_k$ are functions of the initial guess. The expected value of this estimator is

$$E[\hat{x}_k] = x + E[P_k H_k^T R_k^{-1}(v_k + b_E)] \tag{52}$$

In general, the second term in Eq. (52) will not be zero and, as a result, the estimator is biased. The bias is dependent on the quality of the approximation made in Eq. (44) and the linearization made in calculating the covariance and gradient matrices.

The IEKF can also be derived from the cost function in Eq. (43). Instead of expanding $F_k$ about the initial guess, it is expanded about the previous iteration's estimate

$$F_k(x) \approx F_k(\hat{x}_{k,i}) + \left.\frac{\partial F_k}{\partial x}\right|_{x=\hat{x}_{k,i}} (x - \hat{x}_{k,i}) \tag{53}$$

Substituting this into Eq. (43) gives the cost function to be minimized at each iteration

$$J_{k,i} = \frac{1}{2}[(x-\bar{x}_k)^T M_k^{-1}(x-\bar{x}_k) + (z_{k,i}' - H_{k,i} x)^T R_k^{-1}(z_{k,i}' - H_{k,i} x)] \tag{54}$$

where

$$z_{k,i}' = z_k - F_k(\hat{x}_{k,i}) + H_{k,i}\hat{x}_{k,i} \tag{55}$$

and

$$H_{k,i} = \left.\frac{\partial F_k}{\partial x}\right|_{x=\hat{x}_{k,i}} \tag{56}$$

Minimizing this cost function yields iterative equations for determining the measurement update

$$\begin{aligned} \hat{x}_{k,i+1} &= \bar{x}_k + P_{k,i} H_{k,i}^T R_k^{-1}(z_{k,i}' - H_{k,i}\bar{x}_k) \\ P_{k,i} &= (M_k^{-1} + H_{k,i}^T R_k^{-1} H_{k,i})^{-1} \end{aligned} \tag{57}$$

Again, these equations are the same as those derived in Ref. 7. It can be shown that the IEKF measurement update is equivalent to a Gauss-Newton method.[14] This means that iteration of the above equations will cause the approximate cost function in Eq. (54) to approach the cost function in Eq. (43) as $\hat{x}_{k,i+1} \to \hat{x}_{k,i}$. Thus the IEKF obtains the optimal estimate for each time step. However, the covariance matrix, which depends on the current estimate, is approximate. At each succeeding time step, the previous iteration's estimate and covariance matrix are used as the new *a priori* information. However, this information is approximate, resulting in an overall cost function that differs from the original cost function in Eq. (6) and, hence, a suboptimal estimate. This illustrates a

fundamental difference between the IEKF and the two-step estimator, both of which minimize the correct cost function at each time step. The difference is that the two-step estimator stores the information obtained from the previous measurements in the first-step states, which enter the measurement equation in a linear manner. The use of these first-step estimates as the *a priori* information for the next time step allows the two-step estimator to make better use of all the measurements.

The IEKF estimator can also be shown, in a similar manner to that used above for the EKF, to be biased. Again, the approximate Taylor series expansion, Eq. (53), can be made into an equality by adding an error term $b_l$ to the right side. The new equality can be written as

$$b_l = F_k(x) - F_k(\hat{x}_{k,i}) - \left.\frac{\partial F_k}{\partial x}\right|_{x=\hat{x}_{k,i}} (x - \hat{x}_{k,i}) \qquad (58)$$

Using this fact and similar steps to those in the case of the EKF, the expression obtained for the state measurement update is

$$\hat{x}_{k,i+1} = x + P_{k,i} H_{k,i}^T R_k^{-1} (v_k + b_l) \qquad (59)$$

where the covariance matrix $P_{k,i}$ and the gradient matrix $H_{k,i}$ are functions of the previous iteration's estimate. Taking the expected value of the estimator yields

$$E[\hat{x}_{k,i+1}] = x + E[P_{k,i} H_{k,i}^T R_k^{-1} (v_k + b_l)] \qquad (60)$$

The estimator is biased because the second term in Eq. (60), which is a function of the previous iteration's estimate and the error in the approximation of Eq. (53), will not, in general, be zero.

The above derivations show that the EKF and the IEKF are suboptimal estimators for both the static and dynamic problems because they modify the original cost function. By comparison, the two-step method has been shown to be an optimal estimator for the static problem with separable time variation. When the time variation is not separable, the two-step method approaches optimality as the quality of the time update approximation improves. In both cases, the EKF and the IEKF make the approximation of the nonlinearity in the measurement update. In contrast, the two-step estimator moves the approximation to the time update and treats it as a perturbation of the dynamics, thus making better use of the measurements. This results in better estimates than those given by the EKF and the IEKF for many problems, especially those in which most of the information is found in the measurements or very little *a priori* information is available.

The EKF and the IEKF have also been shown to be biased estimators. The EKF is biased and suboptimal overall and at each step because it minimizes a linearized cost function. The IEKF is optimal at each step because it iteratively approaches the correct cost function. However, these IEKF state estimates at each step are biased due to the inherent nonlinearity of the measurements, which transforms the Gaussian probability density function so that minimization of the quadratic cost function no longer yields an unbiased estimator (as it did in the linear case). This is simply a restatement of the fact that the least-squares estimate is not equivalent to the minimum-variance solution for nonlinear problems. By the same reasoning, the second step of the two-step estimator is also biased. The two-step method should, however, result in smaller biases because the biased estimate of the IEKF and its associated approximate covariance matrix are used in the succeeding time steps, whereas the second step of the two-step estimator is done separately from the optimal, unbiased first step in the static problem. Thus the errors in the IEKF get compounded with each step, while the bias of the two-step estimator decreases rapidly as the number of measurements increases. In the case where the time variation is not separable, the biased second-step estimates of the two-step method must be used in the succeeding time steps, but because the nonlinearity is treated as a perturbation of the dynamics, the effect should not be as great.

One common method of dealing with the bias in the EKF and the IEKF has been to artificially increase the state covariance matrix with "fictional" process noise. This increases the filter gains, preventing the filter from "going to sleep" and allowing the estimate to move away from the biased value, hopefully toward the optimal value. Unfortunately, this method is imprecise, requiring tuning of the process noise, and it tends to increase the estimate errors relative to the optimal solution.

The compounding of the error at each time step in the IEKF illustrates that the two-step method should outperform the IEKF in problems where the states become observable over time due to variation in the measurement matrix. This is because the first step of the two-step estimator is linear, so it optimally uses all the measurements even though the states are not observable at any given time step. The IEKF, however, makes a linearization at each time step using only the *a priori* information and the current measurement. The linearization can be poor and the IEKF can get a bad estimate, which is then fed into the next measurement update. The static example given in the next section is a problem in which the time variation of the

measurement matrix is used to make the states observable.

Estimate optimality and bias are not, of course, the only criteria on which to judge an estimation algorithm. Another important criterion is speed. The computation time of the two-step estimator is similar to that of the IEKF, but both are slower than the EKF. Some studies have shown, however, that most of the benefit of iteration is gained in the first or second iterations.[15] Limiting the number of iterations performed at each time step of the two-step method may make its computation time more comparable to that of the EKF, while retaining most of the optimality. Robustness to unmodeled effects is also an important criterion, but it can be difficult to define because it is very problem dependent. The comparisons made in this paper assume perfect measurement and dynamic models.

Static Example—GP-B Data Reduction

The example chosen for the static case is based on the Gravity Probe B Relativity Mission (GP-B) data reduction problem. GP-B is a space experiment intended to test Einstein's General Theory of Relativity by measuring tiny changes in the direction of the spin axis of an Earth-orbiting gyroscope.[16] The measurements taken during the experiment are proportional to the angle that the spin axis makes with respect to an inertial reference direction. This signal is modulated by rolling the satellite about the reference direction. A simplified model of the measurements that will be taken during the experiment[17] is

$$z_k = c(at_k + b + \lambda \sin(\omega_1 t_k))\cos(\omega_2 t_k + d) + v_k \quad (61)$$

where $t$ (time), $\lambda$, $\omega_1$ and $\omega_2$ are known. The measured angle consists of a relativistic drift rate $a$, an initial bias angle $b$, and a sinusoidal signal of amplitude $\lambda$ due to a known variation of the reference direction at the satellite's orbital frequency $\omega_1$. This angle gets modulated at the satellite roll frequency $\omega_2$, which has an unknown phase bias $d$, and then gets multiplied by the unknown scale factor $c$ of the measuring device, which converts the angle in arcseconds to a voltage. The constant unknown states to be estimated are

$$x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad (62)$$

Using a trigonometric identity to substitute for the cosine term, this measurement equation can be rewritten as

$$z_k = H_k y + v_k \quad (63)$$

where

$$H_k^T = \begin{bmatrix} t_k \cos(\omega_2 t_k) \\ -t_k \sin(\omega_2 t_k) \\ \cos(\omega_2 t_k) \\ -\sin(\omega_2 t_k) \\ \lambda \sin(\omega_1 t_k)\cos(\omega_2 t_k) \\ -\lambda \sin(\omega_1 t_k)\sin(\omega_2 t_k) \end{bmatrix} \quad (64)$$

and

$$y = f(x) = \begin{bmatrix} ca\cos(d) \\ ca\sin(d) \\ cb\cos(d) \\ cb\sin(d) \\ c\cos(d) \\ c\sin(d) \end{bmatrix} \quad (65)$$

These equations can now be used directly in the two-step estimator.

The parameter values used in the simulations are given in Table 1. The measurements for the simulations were generated using Eq. (61). Since this is a static problem, the optimal static form of the two-step estimator is used. The true state is taken to be $\mu_x$, while the initial guess is generated randomly for each Monte Carlo run by assuming a normal distribution with mean $\mu_x$ and covariances given by matrix $M_1$. This provides the *a priori* information for the EKF and

**Table 1 Parameter values for the GP-B problem**

Given:

| | | | |
|---|---|---|---|
| $\omega_1$ | = | $2\pi/6000$ rad/sec | |
| $\omega_2$ | = | $2\pi/180$ rad/sec | |
| $\lambda$ | = | 5 arcsec | |
| $T$ | = | 10 sec | sampling time |
| $R_k$ | = | $9 \times 10^{-4}$ volts$^2$ | meas. noise variance |

States:

$$\mu_x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad M_1 = \begin{bmatrix} 0.001^2 & 0 & 0 & 0 \\ 0 & 0.5^2 & 0 & 0 \\ 0 & 0 & 0.05^2 & 0 \\ 0 & 0 & 0 & 0.01^2 \end{bmatrix}$$

where:

| | | |
|---|---|---|
| $a$ | = | 0.001 arcsec/sec |
| $b$ | = | 10 arcsec |
| $c$ | = | 1 volt/arcsec |
| $d$ | = | 0 rad |

the IEKF. The two-step algorithm assumes a very large initial covariance because it is independent of the initial guess for this problem. The plots which follow are for estimates of the relativistic drift rate parameter $a$. Any of the other unknown states could also have been displayed and would have given similar results.

Figure 1 shows that the two-step estimator gives the optimal estimate while the EKF and the IEKF do not. It also shows that the two-step estimator gives a good prediction of the error associated with the estimates. The root-mean-square (RMS) error in the estimate of the relativistic drift rate is calculated for 100 Monte Carlo runs. The figure shows how this RMS error evolves over time for the different estimators. The two-step estimate matches that of the iterated Gauss-Newton batch fit, which makes optimal use of all the data. The EKF and the IEKF differ significantly from the optimal estimate. The only reason they are even close (and actually a little better at the beginning) is that the initial guess was very good. With less *a priori* information, the EKF and the IEKF often diverged, yielding very poor estimates. In contrast, the two-step estimator is independent of the initial guess for this static problem. The predicted covariance of the states for the two-step estimator is also plotted, and it agrees very well with the actual RMS error. The EKF and the IEKF were found to give similar covariance predictions, but these do not reflect the true errors in the estimates and are an indication that the estimators have become biased.

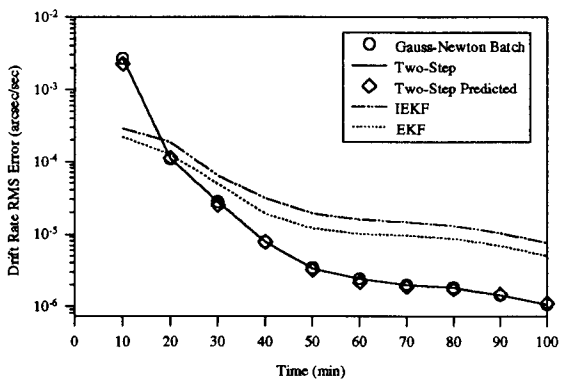


**Fig. 1 Relativistic drift rate RMS error for 100 Monte Carlo runs.**

Figure 2 shows that the two-step estimator is less biased than either the EKF or the IEKF. For an estimator to be unbiased, it must have the property that $E[\hat{x}] = x$. To determine the expected value of the estimator from simulation, a running average of the estimate is plotted over many Monte Carlo runs to show how it converges toward a final value. The EKF and the IEKF clearly have larger biases than the two-step estimator, which appears to be nearly unbiased. Again, it should be noted that the biases of the EKF and the IEKF are very dependent on the initial guess, which was very good in these runs.

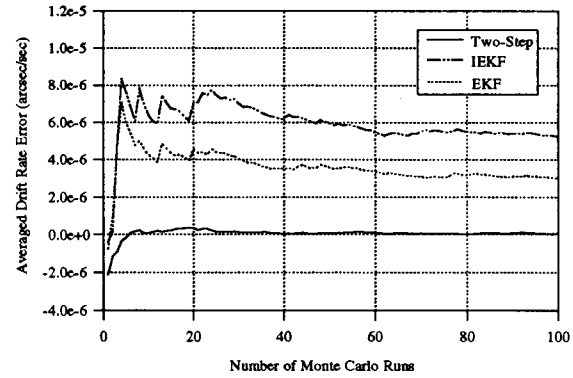


**Fig. 2 Bias in estimate of relativistic drift rate for a 100-minute simulation.**

Dynamic Example—Radar Ranging

The example chosen for the dynamic problem is that of estimating position and velocity of an airplane flying overhead at constant altitude using measurements of the distance to the plane from a radar station on the ground. This is illustrated in Fig. 3, from which the state vector can be defined as

$$x = \begin{bmatrix} a \\ \dot{a} \\ b \end{bmatrix} \tag{66}$$

where $a$ is the horizontal range of the plane, $\dot{a}$ is the horizontal velocity, and $b$ is the altitude. The measurement equation is

$$z(t) = \sqrt{a^2 + b^2} + v \tag{67}$$

and a kinematic model of the state dynamics can be expressed as

$$\dot{x} = Fx + Gw = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} w \tag{68}$$

where a random walk has been added to the horizontal velocity. The continuous state dynamics can be converted to the following discrete dynamic equations[18]

$$x_{k+1} = \Phi x_k + \Gamma w_k = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} T^2/2 \\ T \\ 0 \end{bmatrix} w_k \tag{69}$$

where $T$ is the sampling period. The measurement at time step $k$ can be written

$$z_k = \sqrt{a_k^2 + b^2} + v_k \tag{70}$$

In this example, the second-step states are not observable from the measurements. However, the method described earlier in the section on choosing the first-step states can be used to solve this problem. Following Eq. (39), the first-step states $y$ can be chosen as

$$y_k = f(x_k) = \begin{bmatrix} \sqrt{a_k^2 + b^2} \\ a_k \\ \dot{a}_k \\ b \end{bmatrix} \tag{71}$$

where the unknown state vector $x$ has been appended to the measurement function. The new measurement equation is

$$z_k = Hy_k + v_k = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} y_k + v_k \tag{72}$$

Equations (71) and (72) can now be used in the two-step estimator.

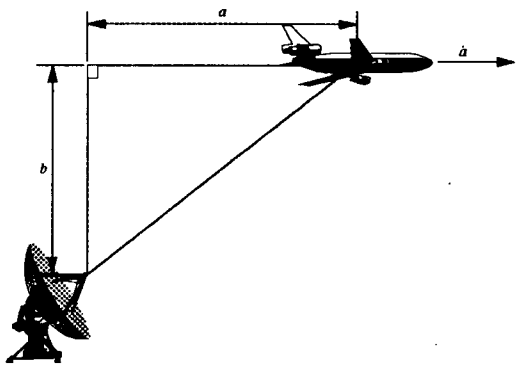


**Fig. 3 Radar ranging of an airplane flying overhead at a constant altitude.**

The parameter values used in the simulations are given in Table 2. The measurements for the simulations were generated using Eqs. (69) and (70). The true state is taken to be $\mu_x$, while the initial guess is generated randomly for each Monte Carlo run by assuming a normal distribution with mean $\mu_x$ and covariances given by matrix $M_1$. This provides the *a priori* information for the EKF, IEKF and two-step algorithms. For the two-step method, the initial guess for the first state in Eq. (71) is derived from the other states. The plots which follow are for estimates of the horizontal range parameter $a$. Any of the other unknown states could also have been displayed and would have given similar results.

Figure 4 shows that the two-step estimator nearly achieves the optimal estimate while the EKF and the IEKF do not. The RMS error in the estimate of the horizontal range is calculated for 100 Monte Carlo runs. The figure shows how this error evolves over time for the different estimators. The two-step estimate is close to the estimate given by an iterated smoother like that found in Ref. 4, which makes optimal use of all the data. The EKF and the IEKF differ significantly from the optimal estimate and are highly dependent upon the initial guess. In fact, the EKF is divergent, with no improvement in the estimate error as measurements are added. In contrast, the two-step estimator is nearly independent of the initial guess. The predicted covariance of the states for the two-step estimator is seen to underpredict the actual RMS error. Similar results were found for the EKF and the IEKF (not shown). This underprediction of the actual error is an indication that the estimators have become biased. The bias in the two-step estimator was found to decrease greatly as the measurement noise was reduced. This supports the notion that the two-step algorithm works well for problems in which most of the information is contained in the measurement equation.

**Table 2 Parameter values for the radar ranging problem**

Given:

| | | | |
|---|---|---|---|
| $T$ | = | 0.1 sec | sampling time |
| $R_k$ | = | 1 ft$^2$ | meas. noise variance |
| $Q_k$ | = | 1 ft$^2$ / sec$^4$ | process noise variance |

States:

$$\mu_x = \begin{bmatrix} a \\ \dot{a} \\ b \end{bmatrix} \quad M_1 = \begin{bmatrix} 100^2 & 0 & 0 \\ 0 & 100^2 & 0 \\ 0 & 0 & 100^2 \end{bmatrix}$$

where:

| | | |
|---|---|---|
| $a$ | = | 1000 ft |
| $\dot{a}$ | = | 500 ft/sec |
| $b$ | = | 3000 ft |

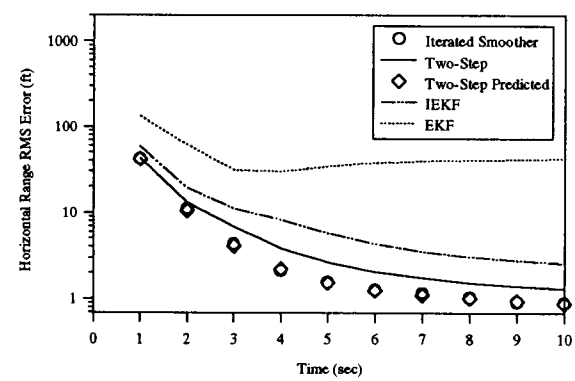


**Fig. 4 Horizontal range RMS error for 100 Monte Carlo runs.**

Figure 5 shows that the two-step estimator is less biased than the IEKF. For a dynamic estimator to be unbiased, it must have the property that $E[\hat{x}_k] = x_k$. To determine the expected value of the estimator from simulation, a running average of the estimate is plotted over many Monte Carlo runs to show how it converges toward a final value. The IEKF clearly has a larger bias than the two-step estimator, which approaches the optimal iterated smoother. The bias of the EKF was much larger than those plotted in Fig. 5 and is not shown. Again, it should be noted that the biases of the EKF and the IEKF are very dependent on the initial guess, which was very good in these runs.

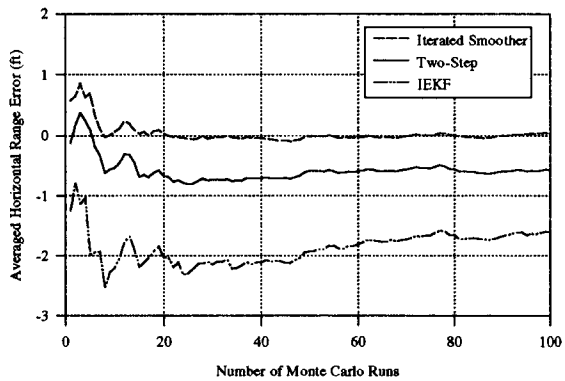


**Fig. 5 Bias in estimate of horizontal range for a 10-second simulation.**

## Conclusions

A new estimation scheme, the two-step estimator, was developed for problems with discrete, nonlinear measurements. The motivation for developing this new algorithm was to devise a recursive estimator which is guaranteed to minimize the desired quadratic cost function. The two-step estimator achieves optimality with a recursive formulation by splitting the cost function into two parts, a linear first step and a nonlinear second step. In contrast to the suboptimal EKF and IEKF, the two-step algorithm was shown to obtain the optimal estimate for problems in which the time variation can be separated from the unknowns. While all three give biased estimates because of their least-squares formulation, the bias of the two-step estimator decreases rapidly as the number of measurements increases. In the case of inseparable time variation, an approximation of the nonlinearity was made which allowed the second-step dynamics to be treated as a perturbation of the first-step dynamics. Though this approximation makes the two-step estimator suboptimal, it was argued that the approximation in the two-step algorithm is better than that of the EKF or the IEKF because the nonlinearity is treated as a small dynamic perturbation of the time update, rather than as a linearization of the measurement equation. Two example applications were used to demonstrate the effectiveness of the two-step method. The static example, taken from the GP-B data reduction problem, showed nearly exact agreement with the optimal Gauss-Newton batch fit. The dynamic example, a radar ranging problem, showed very good agreement with the optimal iterated smoother. In both cases, the two-step estimator performed considerably better than either the EKF or the IEKF.

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