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## Gravity Probe B Relativity Mission

# GP-B Rotor Mass Unbalance And Difference in Moments of Inertia

(Update for Gyro #4 Replacement)

## S0288, Rev. B

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**ITAR Control Req'd?** ☐ Yes ☒ No

# GP-B Rotor Mass Unbalance and Difference in Moments of Inertia S0288

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Change History:

Rev. B –

- Document Updated to Reflect Replacement of Gyroscope #4.
- Summary of Properties of Flight Gyroscopes Added at End of Document

Introduction:

The purpose of this document is to verify the requirements on the mass unbalance and the moment of inertia of the GP-B rotors. The requirements in the SIA specification are

**3.7.1.5.2.2.1.3 Mass Unbalance**

The gyro mass unbalance shall meet the requirements of Section 1.3.2 of T003.

The System Design and Performance Requirements, T003, in turn, gives the requirements on the mass unbalance and moments of inertia as

**1.3.1 Moment of Inertia**

The non-spinning coated rotor shall have a moment-of-inertia difference ratio (delta-I/I) which does not exceed 1e-5 in any axis

**1.3.2 Mass Unbalance**

The coated rotors shall have a total mass unbalance less than 50 nm (2 microinches) (goal = 25 nm)

There are no requirements on the moment of inertia in the SIA specification.

Contributions to Mass Unbalance:

The mass unbalance, $\vec{a}$, of a rotor is defined as the distance between its center-of-mass and its center-of-geometry.

$$\vec{a} = \vec{r}_{cg} - \vec{r}_{cm} \tag{1}$$

where the center-of-geometry is defined as the center of a best fit sphere to the rotor's surface,

$$\vec{r}_{cg} = \frac{1}{S} \int_{Surface} dS\, \vec{r}, \tag{2}$$

and the center-of-mass is defined by the equation

$$\vec{r}_{cm} = \frac{1}{M} \int_{Mass} dM\, \vec{r} = \frac{1}{M} \int_{Volume} dV \vec{r} \rho(r). \tag{3}$$

The three contributions to the mass unbalance of the coated rotor are from the mass unbalance of the coating, the density inhomogeneity of the substrate, and defects in the coated surface. There is no contribution to the mass unbalance due to the asphericity of the substrate.

The contribution to the mass unbalance due to the coating may be calculated in a coordinate system where the origin lies at the center-of-geometry of the uncoated rotor. In this coordinate system, the center-of-geometry of the coated sphere is given by

$$\vec{r}_{cg} = \vec{t} \tag{4}$$

where $\vec{t}$ is a vector whose components are the three dipole components of the coating thickness. The center of mass of the coating is given by

$$\vec{r}_{cm} = \frac{\rho_c}{M} \int_{Coating} dV\vec{r} = \frac{\rho_c \vec{t}}{\rho_s} \tag{5}$$

where $\rho_c$ is the density of the coating and $\rho_s$ is the average density of the substrate. Then, the contribution to the mass unbalance from the coating is

$$\vec{a} = \vec{r}_{cg} - \vec{r}_{cm} = \left( \frac{\rho_c}{\rho_0} - 1 \right) \vec{t} \tag{6}$$

The thickness of the coating is measured by a beta-ray backscattering technique using GP-B procedure P0056. A summary of the mass unbalance due to the coating for each of the flight rotors is included in Table I. Also included in Table I is an estimated error based on the standard deviation of a set of 4 to 6 measurements.

The mass unbalance from the density inhomogeneity of the substrate may be calculated in the same coordinate system. In this coordinate system, the center-of-geometry lies at the origin, and the center of mass is given by

$$\vec{r}_{cm} = \frac{1}{M} \int_{Substrate} dV\vec{r}\rho(\vec{r}) \tag{7}$$

The density may be expanded in a Taylor series about the center of geometry of the rotor,

$$\rho(\vec{r}) = \rho_0 + \vec{r} \cdot \vec{\nabla}\rho\Big|_0 + \frac{1}{2}\vec{r} \cdot \vec{\nabla}\vec{\nabla}\rho\Big|_0 \cdot \vec{r} + \ldots \tag{8}$$

Substituting this expansion into equation (7),

$$\vec{a} = \vec{r}_{cm} = -\frac{r_0^{\,2}}{5\rho_0} \vec{\nabla}\rho\Big|_0 \tag{9}$$

Estimates of the contribution of the density inhomogeneity to the mass unbalance may be made by assuming that the gradient in the density is equal to the maximum variation in the observed density, $\Delta\rho$, divided by the diameter of the rotor. Then, the contribution to the mass unbalance from the density inhomogeneity is

$$|\vec{a}|_{in\hom ogeneity} = \frac{r_0}{10} \frac{\Delta\rho}{\rho} \tag{10}$$

A density inhomogeneity of $10^{-5}$ (SIA Specification 3.7.1.5.2.2.1.1.2) will give a contribution to the mass unbalance of 19 nm (0.76 μin.), which may be compared to the overall requirement of 50 nm (2 μin.). A summary of the methods of verifying the density inhomogeneity is given in GP-B document S0287. For the fused quartz rotors the density inhomogeneity of each of the fused quartz cubes was measured. The measured density inhomogeneity and the corresponding maximum contribution to the mass unbalance is listed in Table I. The estimated error for the contribution to the mass unbalance from the density inhomogeneity of the fused quartz spheres is 0.5nm (0.02 μin.). For the single crystal silicon spheres, the density inhomogeneity is conservatively estimated to be less than $10^{-6}$. With this density inhomogeneity, the maximum contribution to the mass unbalance is 1.9 nm (0.08 μin.) with an error of 1.9 nm (0.08 μin.) .

The additional contribution to the mass unbalance from any defects in the surface is given by

$$a = \frac{r_0 \rho A t}{M} \tag{11}$$

where the missing material has area A, density ρ, and thickness t, $r_0$ is the radius of the rotor, and M is the total mass of the rotor. A worst case mass unbalance due to 5 patches having a diameter of 0.25 mm and a depth equal the thickness of the coating 1.25 microns is 0.08 nm. Since this mass unbalance is negligible compared to the other two contributions, it is not included in Table I.

Table I
Estimated Contributions to Mass Unbalance

| | | | Worst-on-worst Total | |
|---|---|---|---|---|
| Rotor | Coating (μin.) | Substrate (μin.) | (μin.) | (nm) |
| 95FH1 | 0.28 ± 0.07 | 0.19 ± 0.02 | 0.47 ± 0.07 | 11.9 ± 1.8 |
| 95FH3 | 0.35 ± 0.10 | 0.17 ± 0.02 | 0.53 ± 0.10 | 13.5 ± 2.5 |
| 95FH4 | 0.43 ± 0.12 | 0.17 ± 0.02 | 0.60 ± 0.12 | 15.2 ± 3.1 |
| 95FH5 | 0.50 ± 0.23 | 0.26 ± 0.02 | 0.76 ± 0.23 | 19.3 ± 5.8 |
| 95FH6 | 0.35 ± 0.14 | 0.22 ± 0.02 | 0.57 ± 0.14 | 14.5 ± 3.6 |
| 95FH7 | stripped | 0.21 ± 0.02 | - | - |
| 95FH9 | 0.45 ± 0.09 | 0.21 ± 0.02 | 0.66 ± 0.09 | 16.8 ± 2.3 |
| 95FH10 | 0.43 ± 0.09 | 0.21 ± 0.02 | 0.64 ± 0.09 | 16.3 ± 2.3 |
| 96FH11† | 0.375 ± 0.10 | 0.23 ± 0.02 | 0.61 ± 0.10 | 15.5 ± 2.5 |
| 96FH12 | 0.32 ± 0.07 | 0.30 ± 0.02 | 0.62 ± 0.07 | 15.8 ± 1.8 |
| 96FH13 | stripped | 0.30 ± 0.02 | - | - |
| 96FH14 | 0.27 ± 0.15 | 0.26 ± 0.02 | 0.53 ± 0.15 | 13.5 ± 2.3 |
| 96FH15 | 1.34 ± 0.18 | 0.23 ± 0.02 | 1.57 ± 0.18 | 39.9 ± 4.6 |
| 96FH16 | 0.41 ± 0.17 | 0.26 ± 0.02 | 0.67 ± 0.17 | 17.0 ± 4.3 |
| 96FH17 | 0.50 ± 0.14 | 0.24 ± 0.02 | 0.74 ± 0.14 | 18.8 ± 3.6 |
| 96FH18 | 0.51 ± 0.12 | 0.28 ± 0.02 | 0.79 ± 0.12 | 20.1 ± 3.1 |
| 93S17 | 1.77 ± 0.17 | 0.08 ± 0.08 | 1.85 ± 0.17 | 47.0 ± 4.3 |
| 93S24 | 1.74 ± 0.17 | 0.08 ± 0.08 | 1.82 ± 0.17 | 46.2 ± 4.3 |
| 93S25 | 1.73 ± 0.06 | 0.08 ± 0.08 | 1.81 ± 0.10 | 46.0 ± 2.5 |
| 95S26 | 0.90 ± 0.36 | 0.08 ± 0.08 | 0.98 ± 0.36 | 24.9 ± 9.1 |
| 96S27 | 0.69 ± 0.26 | 0.08 ± 0.08 | 0.77 ± 0.27 | 19.6 ± 6.9 |
| 96S29 | 0.49 ± 0.13 | 0.08 ± 0.08 | 0.57 ± 0.15 | 14.5 ± 3.8 |
| 96 S30 | 1.32 ± 0.11 | 0.08 ± 0.08 | 1.40 ± 0.14 | 35.6 ± 3.6 |
| 97S31 | 0.60 ± 0.19 | 0.08 ± 0.08 | 0.68 ± 0.21 | 17.3 ± 5.3 |

† Recoated 9/98

Contribution to the Difference in Moments of Inertia

The moment of inertia tensor may always be transformed to a coordinate system where the moment of inertia tensor is diagonal. In this coordinate system, the three principal moments of inertia are

$$I_{ii} = \int_{Volume} dV \rho(\vec{r})\left(r^2 - x_i^2\right) \qquad (12)$$

and the three differences in the principal moments of inertia are given by

$$\Delta I_{ij} = I_{ii} - I_{jj} = \int_{Volume} dV \rho(\vec{r})\left(x_j^2 - x_i^2\right) \qquad (13)$$

Since the principal axes of each of the contributions is not necessarily along the same principal axes, the maximum difference in the moments of inertia is less than the maximum difference in the moments of inertia of the contributions. Dividing the difference in moments of inertia into the contribution from the coating, the contribution from the asphericity of the substrate, the contribution from the density inhomogeneity in the substrate, and any contributions from defects in the coating:

$$\Delta I_{\max} \leq \Delta I_{coating} + \Delta I_{asphericity} + \Delta I_{in\hom ogeneity} + \Delta I_{defects} \qquad (14)$$

where each of the contribution to the differences in moment of inertia is taken as the maximum difference in the three moments of inertia.

The contribution to the difference in the moments of inertia from the coating is found by calculating the three principal moments of inertia from the coating thickness measurements and finding the maximum difference in these moments of inertia. The maximum values for the fractional difference in the moments of inertia due to the coating are listed in Table II. The error in each one of these differences in the moments of inertia is the standard deviation of 4 to 6 measurements.

For a uniform density rotor with a radius given by

$$r = r_0 + r_2 \frac{x^2 + y^2}{r_0^{\,2}}, \qquad (15)$$

the fractional difference in the moments of inertia due to this asphericity is given by

$$\frac{\Delta I_{asphericity}}{I} = \frac{\rho_0}{I} \int_{Surface} (x^2 - z^2) r dS = \frac{r_2}{r_0} \qquad (16)$$

A peak-to-valley quadrupole component of the asphericity of $r_2$=25 nm (1 μin.) gives a fractional difference in the moments of inertia of $\Delta I/I = 1.3 \times 10^{-6}$. The contribution to the fractional difference in the moments of inertia based on the measured peak-to-valley L=2 (quadrupole) component of the asphericity of the uncoated rotor are listed in Table 2 for each of the flight rotors. The error associated with each of these contributions to the difference in moments of inertia is estimated to be $0.2 \times 10^{-6}$.

Similarly, if the density of the rotor varies radially

$$\rho = \rho_0 + \frac{x^2 + y^2}{r_0^{\,2}} \rho_2 \qquad (17)$$

then the fractional difference in the moment of inertia is given by

$$\frac{\Delta I_{in\,\hom ogeneity}}{I} = \frac{I_{xx} - I_{zz}}{I} = \frac{1}{I} \int_{Volume} \rho\left(z^2 - x^2\right) dV = \frac{\rho_2}{7\rho_0} \tag{18}$$

If $\rho_2/\rho_0 = 10^{-5}$, the $\Delta I/I = 1.4 \times 10^{-6}$. An upper limit on the difference in moments of inertia due to the density inhomogeneity of the substrate is given in Table II. For the fused quartz rotors, these estimates are based on the measured density inhomogeneity with the errors are due to the error in the measurement of the density inhomogeneity. For the single crystal silicon rotors, only upper limits on the density inhomogeneity exist, so the error in the difference of the moments of inertia is equal to its estimated value.

A pit in the substrate or a missing patch in the rotor coating having an area, A, and a depth, d, will give a contribution to the fractional difference in the moments of inertia of

$$\frac{\Delta I_{defects}}{I} = \frac{\rho r_0^{\,2} A\, d}{I} \tag{19}$$

Five missing patches in the rotor coating having a diameter of 0.25 mm and a depth equal to the thickness of the coating, 1.25 microns, would give a contribution to the fractional differences in the moments of inertia of $2.5 \times 10^{-8}$. This contribution is small enough that it may be ignored.

The contributions to the fractional differences in the moments of inertia are summarized in Table II. The contributions from the density inhomogeneity are upper limit on this contribution. Since the principal axes of each one of the contributions are unlikely to be aligned, the best estimate of the fractional differences in the moments of inertia are given by the root-square sum.

Table II
Estimated Contributions to Fractional Difference in Moments of Inertia

| | Contribution to Fractional Difference in Moment of Inertia, $\Delta I/I$, $\times 10^{-6}$ | | | |
|---|---|---|---|---|
| Rotor | Coating | Substrate Asphericity | Substrate Inhomogeneity | Worst-on-worst Total |
| 95FH1 | 2.6 ± 1.0 | 0.5 ± 0.2 | 0.4 ± 0.05 | 3.5 ± 1.0 |
| 95FH3 | 2.0 ± 0.6 | 0.6 ± 0.2 | 0.3 ± 0.05 | 2.9 ± 0.6 |
| 95FH4 | 2.3 ± 0.6 | 1.4 ± 0.2 | 0.3 ± 0.05 | 4.0 ± 0.6 |
| 95FH5 | 2.6 ± 1.3 | 1.1 ± 0.2 | 0.5 ± 0.05 | 4.2 ± 1.3 |
| 95FH6 | 2.0 ± 0.9 | 0.9 ± 0.2 | 0.4 ± 0.05 | 3.3 ± 0.9 |
| 95FH7 | stripped | 0.7 ± 0.2 | 0.4 ± 0.05 | - |
| 95FH9 | 2.2 ± 0.7 | 1.0 ± 0.2 | 0.4 ± 0.05 | 3.6 ± 0.7 |
| 95FH10 | 1.3 ± 0.3 | 0.7 ± 0.2 | 0.4 ± 0.05 | 2.4 ± 0.4 |
| 96FH11† | 6.1 ± 2.0 | 0.7 ± 0.2 | 0.4 ± 0.05 | 7.2 ± 2.0 |
| 96FH12 | 2.6 ± 0.3 | 0.6 ± 0.2 | 0.6 ± 0.05 | 3.8 ± 0.4 |
| 96FH13 | stripped | 0.3 ± 0.2 | 0.6 ± 0.05 | - |
| 96FH14 | 2.6 ± 0.4 | 0.9 ± 0.2 | 0.5 ± 0.05 | 4.0 ± 0.5 |
| 96FH15 | 3.3 ± 1.1 | 0.6 ± 0.2 | 0.4 ± 0.05 | 4.3 ± 1.1 |
| 96FH16 | 2.3 ± 0.9 | 0.8 ± 0.2 | 0.5 ± 0.05 | 3.6 ± 0.9 |
| 96FH17 | 2.9 ± 1.2 | 0.9 ± 0.2 | 0.5 ± 0.05 | 4.3 ± 1.2 |
| 96FH18 | 2.4 ± 0.7 | 0.4 ± 0.2 | 0.5 ± 0.05 | 3.3 ± 0.7 |
| 93S17 | 2.7 ± 0.8 | 0.2 ± 0.2 | 0.1 ± 0.1 | 3.0 ± 0.8 |
| 93S24 | 2.4 ± 0.9 | 0.2 ± 0.2 | 0.1 ± 0.1 | 2.7 ± 0.9 |
| 93S25 | 2.9 ± 1.1 | 0.7 ± 0.2 | 0.1 ± 0.1 | 3.7 ± 1.1 |
| 95S26 | 3.0 ± 0.3 | 0.6 ± 0.2 | 0.1 ± 0.1 | 3.7 ± 0.4 |
| 96S27 | 4.8 ± 0.6 | 0.4 ± 0.2 | 0.1 ± 0.1 | 5.3 ± 0.6 |
| 96S29 | 4.5 ± 0.4 | 0.3 ± 0.2 | 0.1 ± 0.1 | 4.9 ± 0.5 |
| 96 S30 | 3.7 ± 1.2 | 0.3 ± 0.2 | 0.1 ± 0.1 | 4.1 ± 1.2 |
| 97S31 | 3.6 ± 1.5 | 0.4 ± 0.2 | 0.1 ± 0.1 | 4.1 ± 1.5 |

† Recoated 9/98

Summary

The table below summarizes the mass unbalance and fractional differences in moments of inertial for the four flight gyroscopes:

| Gyroscope | | | Mass Unbalance | | |
|---|---|---|---|---|---|
| # | Housing | Rotor | (μin.) | (nm) | $\Delta I/I$, $\times 10^{-6}$ |
| 1 | FQH61 | 96FH17 | 0.74 ± 0.14 | 18.8 ± 3.6 | 4.3 ± 1.2 |
| 2 | FQH46 | 95FH06 | 0.57 ± 0.14 | 14.5 ± 3.6 | 3.3 ± 0.9 |
| 3 | FQH44 | 96FH09 | 0.66 ± 0.09 | 16.8 ± 2.3 | 3.6 ± 0.7 |
| 4 | FQH58 | 95FH03 | 0.53 ± 0.10 | 13.5 ± 2.5 | 2.9 ± 0.6 |

All four gyroscopes are well within the specifications.