W. W. Hansen Experimental Physics Laboratory
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305 - 4085

Gravity Probe B Relativity Mission

ANALYTICAL SOLUTION FOR UNMODELED ERRORS IN THE GRAVITY PROBE B EXPERIMENT

S0895, Rev. -
June, 2003

Prepared by:

6/17/03
G. M. Keiser — Date
GP-B Chief Scientist

Reviewed by:

06/13/2003
M. Heifetz — Date

Approved by:

6/17/2003
R. Whelan — Date
Systems Engineering

ITAR Assessment Performed
Tom Langenstein

6.17.03
ITAR Control Req'd? ☐ Yes ☒ No

Analytical Solution for Unmodeled Errors In the Gravity Probe B Experiment S0895

Table of Contents

A. Introduction

Systematic measurement errors may arise either (1) from effects which have exactly the same time signature as the effects that are being measured or (2) from effects that have a different time signature but nevertheless some correlation with the measured effects. In the first case, experimental techniques may be used to place limits on these errors, but limits may not be placed on these error sources by data analysis alone. In the second case, if the spurious signals are ignored in the data analysis, there will be unmodeled errors in the data analysis, and the post-fit residuals will not be noise limited. As long as the magnitude and time signatures of these unmodeled signals are well understood, and they can be shown to have a negligible effect on the overall experiment error, they can be ignored. For those effects that make a significant contribution to the experiment error, the error bars of the formal statistical error must be increased to account for the potential systematic errors. In some cases, augmenting the data analysis model to include the additional effects will reduce the unmodeled errors. However, this reduction in the unmodeled errors may come at the expense of an increase in the statistical errors of the other measured quantities.

The effects of these unmodeled errors in the Gravity Probe B experiment were first discussed by Haupt [1], who examined the effects of bias error at various frequencies on the gyroscope drift rate. The effects of potential unmodeled data reduction errors have been included in the Gravity Probe B Error Tree [2].. The document, "Gravity Probe B Data Reduction: Analysis of Unmodeled Error", [3] describes two equivalent methods of calculating these unmodeled errors. Using either of these methods, the effect of any assumed temporal variation in the combined gyroscope and telescope signals on the gyroscope drift rate determined from the data analysis may be calculated. These calculations have been done numerically in the Gravity Probe B Error Tree

The purpose of this document is to present the analytical solutions for unmodeled error for various temporal variations in the bias, scale factor, and phase shift of the combined signals from the gyroscopes and telescope in the Gravity Probe B satellite. These analytical solutions provide some insight into the effect of potential disturbances on the experimental results. They also provide a convenient method of verifying some of the numerous requirements on the satellite hardware.

Basic Gravity Probe B Measurement Model

Without any interfering effects, the combined gyroscope and telescope data may be modeled as linear drift in the gyroscope spin axis in two orthogonal directions and the optical effects which cause the apparent direction of the guide star to differ from the true direction. Then, the model for the combined the gyroscope and telescope data, z, may be written as

$$z = C_g \begin{bmatrix} (NS_O + R_{NS}(t-t_0) + a_{NS})\cos(\phi_r + \delta\phi) + \\ + (EW_0 + R_{EW}(t-t_0) + a_{EW})\sin(\phi_r + \delta\phi) + b \end{bmatrix} \tag{A.1}$$

In this equation, z is a scalar function of time, and the symbols have the following meaning: C_g is the scale factor with determines the conversion between the measured voltages and the angle of the gyroscope from the direction to the guide star, and $\delta\phi$ is the angular difference between the measured roll phase, ϕ_r, and the normal to the gyroscope pick-up loop. The angles NS_0 and EW_0 are the components of the orientation of the gyroscope spin axis at time $t = t_0$ in the North-South and East-West directions, R_{NS} and R_{EW} are gyroscope drift rate in these same directions, and a_{NS} and a_{EW} are the time dependent components of the optical effects which cause the apparent direction of the guide star to differ from the true direction to the guide star. These effects include the optical aberration signal due to the orbital motion of the satellite about the earth and the motion of the earth about the sun, parallax, and the gravitational deflection of light by the sun. The parameter b represents an arbitrary bias in the combined gyroscope and telescope signals. The parameters Cg, $\delta\phi$, NS_0, EW_0, R_{NS}, R_{EW}, and b are a minimum set of parameters that will be determined from the data analysis. The time and roll phase are measured.

To determine the scale factor and the roll phase offset, both the magnitude and direction of at least one of the optical effects, which has a unique time signature, must be known with high precision. These optical effects include the aberration of light from the guide star due to the orbital motion of the satellite about the earth and the annual motion of the earth about the sun, the gravitational deflection of light by the sun, the parallax due to the earth's motion about the sun, and, possibly, the orbital motion of the guide star, HR 8703, about its unseen companion at a 24.6 day period. Since the orbital and annual aberration signals are the largest effects (having amplitudes of 5 arc sec and 20 arc sec respectively) and may be measured precisely, these signals serve as known calibration signals which may be used to determine the scale factor and roll phase offset. The magnitude of the other optical effects may either be determined from the data analysis or from other methods. Here, these other optical effects are assumed to be known.

Considerable insight into the measurements may be gained by examining the data over short intervals from several orbits to several days. In this case, all of the optical effects, with the exception of the orbital aberration, are slowly varying functions of time, and are indistinguishable from the average misalignment and drift rate of the gyroscopes. Defining the short term misalignment and drift rate as

$$\begin{aligned} NS_0' &= NS_0 + \bar{a}_{NS} \\ EW_0' &= EW_0 + \bar{a}_{EW} \\ R_{NS}' &= R_{NS} + \left.\frac{\partial a_{NS}}{\partial t}\right|_{t=t_0} \\ R_{EW}' &= R_{EW} + \left.\frac{\partial a_{EW}}{\partial t}\right|_{t=t_0} \end{aligned} \tag{A.2}$$

the basic measurement equation over a short time interval may be rewritten as

$$z = C_g \left[\begin{array}{l} (NS_O' + R_{NS}'(t - t_0) + A_O \cos\phi_O)\cos(\phi_r + \delta\phi) + \\ \qquad + (EW_0' + R_{EW}'(t - t_0))\sin(\phi_r + \delta\phi) + b \end{array} \right] \tag{A.3}$$

Here, A_o is the amplitude of the orbital aberration and ϕ_o is the phase of the orbital motion measured from the closest point to the guide star. Note that this equation is an approximation since the orbital aberration is assumed to depend only on the phase of the orbital motion. While this circular orbit approximation is useful for estimating potential errors, for the actual data analysis the orbital aberration will be determined from the measured position and velocity of the satellite. The other long term optical effects have been absorbed into the parameters NS0', EW0', RNS', and REW'. These parameters NS0', EW0', RNS', and REW' may be considered fixed over any one day interval, but will vary from day to day. This model will not be used for determining the relativistic drift rate. However, it may well be used for the short term data analysis since it should fit the data to within the limits imposed by the gyroscope and telescope readout noise.

Unmodeled Errors

The Gravity Probe B data reduction model is nonlinear. As shown in S0351, "Analytic Solution for the Gravity Probe B Covariance Matrix" [4] and S0354, "Gravity Probe B Data Reduction: Analysis of Unmodeled Error", [3] it may be treated as a two-step data reduction problem where the first step is linear and yields an intermediate set of states, y:

$$z = H\,y + v \tag{A.4}$$

The noise, ν, is assumed to be Gaussian with a standard deviation, σ. The least squares determination of these intermediate states is given by

$$y = P_1 \frac{H^T z}{\sigma^2} \tag{A.5}$$

where these first-step states have the corresponding information and covariance matrices:

$$I_1 = \frac{H^T H}{\sigma^2} \qquad P_1 = I_1^{-1} \tag{A.6}$$

The interesting physical parameters, x, are nonlinear functions of the intermediate states:

$$y = f(x) \tag{A.7}$$

A nonlinear least squares fit may be used to determine the optimum value of the parameters:

$$x = P_2 \left(\frac{\partial f}{\partial x} \right)^T I_1\, y \tag{A.8}$$

where the information and covariance matrices of these second-step states are

$$I_2 = \left(\frac{\partial f}{\partial x}\right)^T I_1 \left(\frac{\partial f}{\partial x}\right) \qquad \text{(A.9)}$$

$$P_2 = I_2^{-1}$$

The covariance matrices, P_1 and P_2, have been calculated analytically. [4].

With this approach for any assumed disturbance in the measurement, Δz, the corresponding unmodeled error in the second step states may be calculated:

$$\Delta x = P_2 \left(\frac{\partial f}{\partial y}\right)^T I_1 \, \Delta y = P_2 \left(\frac{\partial f}{\partial y}\right)^T \frac{H^T \Delta z}{\sigma^2} \qquad \text{(A.10)}$$

where Δz, Δy, and Δx are the unmodeled errors in the data, the first-step states, and the second-step states, respectively.

B. Short Term Data Analysis

For the short term data analysis, the basic model is given by equation (A.3). This equation may be rewritten in the form given by equation (A.4), where the measurement matrix is

$$H = [\cos\phi_r,\ (\frac{t}{t_A})\cos\phi_r,\ \cos\phi_o \cos\phi_r,\ \sin\phi_r,\ (\frac{t}{t_A})\sin\phi_r,\ \cos\phi_o \sin\phi_r] \qquad \text{(B.11)}$$

and the first step states are

$$y = \begin{bmatrix} C_g\left(NS\cos\delta\phi + EW\sin\delta\phi\right) \\ C_g\left(R_{NS}t_A\cos\delta\phi + R_{EW}t_A\sin\delta\phi\right) \\ C_g\left(A_0\cos\delta\phi\right) \\ C_g\left(-NS\sin\delta\phi + EW\cos\delta\phi\right) \\ C_g\left(-R_{NS}t_A\sin\delta\phi + R_{EW}t_A\cos\delta\phi\right) \\ C_g\left(-A_o\sin\delta\phi\right) \end{bmatrix} \qquad \text{(B.1)}$$

The second step states are the scale factor, C_g, the roll phase offset, $\delta\phi$, the misalignments at $t = t_0$, NS_0 and EW_0, and the change in the gyroscope angular orientation due to its drift rate in a one year period in each of the two orthogonal directions, $R_{NS}t_A$ and $R_{EW}t_A$.

For any assumed unmodeled error in the measurements, Δz, the unmodeled errors in the first step and second step states may be calculated using equation (A.10). These calculations may be done numerically, but more insight into the nature of the unmodeled errors may be gained using an analytical solution. Such an analytical solution may be obtained by the straightforward application of equation (A.10) for any assumed unmodeled disturbance in the measurement, Δz. Since the calculation of such an analytical solution is very tedious, the Matlab symbolic math toolbox was employed to calculate the unmodeled first and second step errors for a variety of unmodeled inputs. A copy of this symbolic math program is included in Appendix B.

Fortunately, these calculations may be considerably simplified in the case of the unmodeled errors since they are small perturbations about the basic solution. From equation (B.1), it can be seen that the scale factor, C_g, and roll phase offset, $\delta\phi$, are determined entirely by the first step states, y_3 and y_6. Small perturbations of these first step states give the following perturbations in the scale factor and the roll phase offset:

$$\begin{bmatrix} \Delta y_3 \\ \Delta y_6 \end{bmatrix} = \begin{bmatrix} (\Delta C_g)A_o \cos\delta\phi - (\Delta\delta\phi)C_g A_o \sin\delta\phi \\ -(\Delta C_g)A_o \sin\delta\phi - (\Delta\delta\phi)C_g A_o \cos\delta\phi \end{bmatrix} \tag{B.2}$$

This equation, which is linear in ΔC_g and $\Delta\delta\phi$, may be solved to find the change in the scale factor and roll phase offset for any given change in the first step states, y_3 and y_6:

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \end{bmatrix} = \frac{1}{A_o}\begin{bmatrix} (\cos\delta\phi)\Delta y_3 - (\sin\delta\phi)\Delta y_6 \\ -(\sin\delta\phi / C_g)\Delta y_3 - (\cos\delta\phi / C_g)\Delta y_6 \end{bmatrix} \tag{B.3}$$

Similarly, from equation (A.12) changes in the first step states y_1, y_2, y_4, and y_5 may be found in terms of changes in the second step states. These linear equations may then be solved to find the changes in the second step states in terms of changes in the scale factor, roll phase offset, and these first step states. The result is

$$\begin{bmatrix} \Delta NS \\ \Delta EW \\ \Delta R_{NS}t_A \\ \Delta R_{EW}t_A \end{bmatrix} = \begin{bmatrix} -(\Delta C_g / C_g)NS - (\delta\phi)EW + (\Delta y_1 / C_g)\cos\delta\phi - (\Delta y_4 / C_g)\sin\delta\phi \\ -(\Delta C_g / C_g)EW + (\delta\phi)NS + (\Delta y_1 / C_g)\sin\delta\phi + (\Delta y_4 / C_g)\cos\delta\phi \\ -(\Delta C_g / C_g)R_{NS}t_A - (\delta\phi)R_{EW}t_A + (\Delta y_2 / C_g)\cos\delta\phi - (\Delta y_5 / C_g)\sin\delta\phi \\ -(\Delta C_g / C_g)R_{EW}t_A + (\delta\phi)R_{NS}t_A + (\Delta y_2 / C_g)\sin\delta\phi + (\Delta y_5 / C_g)\cos\delta\phi \end{bmatrix} \tag{B.4}$$

Here, the symbols NS, EW, $R_{NS}t_A$, $R_{EW}t_A$, C_g, and $\delta\phi$ refer to the true values for the second step states while the same symbols preceded by a Δ refer to the perturbed solutions. Equations (A.14) and (A.15) may be combined, of course, to give a single solution to the perturbed positions and drift rates in terms of the perturbations of the first step states. However, keeping the equations separate gives considerably more physical insight into the solution since it explicitly shows which changes in the second step states are due to the changes in the scale factor and the roll phase offset. It also considerably simplifies the expressions for the unmodeled errors. It will be shown in the section on the long term data analysis that equation (A.15) applies to the long term data analysis although equation (A.14) does not.

Bias Variations

Bias Variations at the Satellite Roll Frequency

From equation (A.3), it can be seen that a constant amplitude sinusoidal variation in the bias will be indistinguishable from a misalignment of the gyroscope readout axis relative to the direction to the guide star. This straightforward conclusion is confirmed by the analytical solution of the unmodeled errors for the second step states. The phase of this roll frequency bias variation may coincide the phase due to a misalignment in the north-south direction or in an east-west direction. For a bias variation corresponding to a north-south misalignment,

$$z_u = e_{NS}\cos(\phi_r + \delta\phi) \tag{B.5}$$

The errors in the second-step states are as expected:

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} t_A \\ \Delta R_{EW} t_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e_{NS} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{B.6}$$

Similarly, a bias variation corresponding to a misalignment in the east-west direction

$$z_u = e_{EW}\sin(\phi_r + \delta\phi) \tag{B.7}$$

gives the expected results in the second-step states:

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} t_A \\ \Delta R_{EW} t_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e_{EW} \\ 0 \\ 0 \end{bmatrix} \tag{B.8}$$

These results are the expected results and confirm that the symbolic math program is working as expected. Although these bias variations at the satellite roll frequency will have no direct impact on the determination of the gyroscope drift rate, any variation in the amplitude or phase of these roll frequency variations will produce a systematic error in the measured drift rate. There are no direct requirements on a constant bias variation, but any roll frequency variation is of concern.

Roll Frequency Bias Variations which vary linearly with time

Any bias variation at the satellite roll frequency where the amplitude varies linearly with time will be indistinguishable from the relativistic drift rate of the gyroscope. Therefore, tight requirements are placed on these quantities in the System Design and Performance Requirements, T003. An unmodeled bias variation having the form

$$z_u = e_{RNS} t_A \frac{t}{t_A}\cos(\phi_r + \delta\phi) \tag{B.9}$$

produces the expected errors in the second step states

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} t_A \\ \Delta R_{EW} t_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e_{RNS} t_A \\ 0 \end{bmatrix} \tag{B.10}$$

where e_{RNS} is the angular drift in a period t_A, which is taken to be one year. Similarly, an unmodeled bias variation with the form

$$z_u = e_{REW} t_A \frac{t}{t_A} \sin(\phi_r + \delta\phi) \tag{B.11}$$

produces the corresponding errors in the second step states

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} t_A \\ \Delta R_{EW} t_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ e_{REW} t_A \end{bmatrix} \tag{B.12}$$

Unmodeled errors having this temporal dependence may not be reduced by adding additional states to the basic model since the time dependence is exactly the same as the time dependence of the signal due to the gyroscope drift rate.

Roll Frequency Bias Variation Modulated at the Orbital Frequency

For the short term analysis, the scale factor of the gyroscope readout system and the roll phase offset are determined by from the orbital aberration signal. This signal lies at the satellite roll frequency but its amplitude and phase are modulated at the orbital period. Over longer intervals, calibration of the gyroscope scale factor using this orbital aberration signal is less accurate than the calibration using the annual aberration signal. However, it is nevertheless useful to place constraints on bias variations at this frequency which would limit the accuracy of the scale factor determined from these measurements. An unmodeled error in the combined gyroscope and telescope signals having the same phase as the orbital aberration signal

$$z_u = e_{NSO} \cos\phi_o \cos(\phi_r + \delta\phi) \tag{B.13}$$

produces the error in the second step states

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} t_A \\ \Delta R_{EW} t_A \end{bmatrix} = \begin{bmatrix} (e_{NSO} / A_o) C_g \\ 0 \\ -(\Delta C_g / C_g) NS_o \\ -(\Delta C_g / C_g) EW_0 \\ -(\Delta C_g / C_g) R_{NS} t_A \\ -(\Delta C_g / C_g) R_{EW} t_A \end{bmatrix} \tag{B.14}$$

In this case, an unmodeled error having the same time dependence as the orbital aberration signal, which produces an error in the scale factor of the gyroscope readout. This error in the scale factor, in turn, causes the expected errors in the average gyroscope misalignments and drift rates.

A roll frequency bias variation modulated at the orbital frequency may be 90^0 out of phase with the orbital aberration but have the same dependence on the orbital phase. In this case, the unmodeled error is

$$z_u = e_{EWO}\cos\phi_o \sin(\phi_r + \delta\phi) \quad \text{(B.15)}$$

with the corresponding errors in the second step states:

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} t_A \\ \Delta R_{EW} t_A \end{bmatrix} = \begin{bmatrix} 0 \\ -e_{EWO} / A_o \\ -\Delta\delta\phi\, EW_o \\ \Delta\delta\phi\, NS_0 \\ -\Delta\delta\phi\, R_{EW} t_A \\ \Delta\delta\phi\, R_{NS} t_A \end{bmatrix} \quad \text{(B.16)}$$

Here, the unmodeled error produces an error in the roll phase offset, which, in turn, causes errors in the estimated values for the average gyroscope misalignments and drift rates. Neither of these two unmodeled errors may be reduced by augmenting the basic model since the time dependence is identical to the time dependence of the signal.

Unmodeled errors may also occur at the satellite roll frequency modulated at the orbital frequency but where the modulation at the orbital frequency may be out of phase with the orbital aberration signal. These signals have a dependence on the orbital and roll phase given by

$$\begin{aligned} z_u &= e_{NSP}\sin\phi_o \cos(\phi_r + \delta\phi) \\ z_u &= e_{EWP}\sin\phi_o \sin(\phi_r + \delta\phi) \end{aligned} \quad \text{(B.17)}$$

where e_{NSP} and e_{EWP} are the amplitudes of an unmodeled signal. As long as the data is collected symmetrically about the point in the orbit closest to the guide star, error sources with these time signatures produce no unmodeled errors in the basic six second-step states.

Roll Modulated by Twice Orbital

Some sources of systematic errors in the gyroscope readout, such as leakage of external magnetic field into the gyroscope readout pickup loop, have the potential to produce bias variations at the roll frequency modulated at twice the orbital frequency. Although this time dependence is different than that of the orbital aberration signal, it may nevertheless produce an error in the short term data analysis because data may be taken only during slightly more than half of each orbit. The phase of such a signal will have a significant effect on the unmodeled errors. If the unmodeled signal has the form

$$z_u = e_{b2o}\cos(2\phi_o)\cos(\phi_r + \delta\phi), \quad \text{(B.18)}$$

then there will be unmodeled error in the scale factor

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} t_A \\ \Delta R_{EW} t_A \end{bmatrix} = \begin{bmatrix} e_{b2o}(4\pi/3A_o)/(\pi^2-8) \\ 0 \\ -(\Delta C_g/C_g)(NS_0+2A_0/\pi) \\ -(\Delta C_g/C_g)EW_0 \\ -(\Delta C_g/C_g)R_{NS} \\ -(\Delta C_g/C_g)R_{EW} \end{bmatrix} \tag{B.19}$$

However, if the unmodeled signal has the form

$$z_u = e_{b2o}\cos(2\phi_o)\sin(\phi_r+\delta\phi), \tag{B.20}$$

there will be an unmodeled error in the roll phase offset

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} t_A \\ \Delta R_{EW} t_A \end{bmatrix} = \begin{bmatrix} 0 \\ -(e_{b2o}/C_g)(4\pi/3A_o)/(\pi^2-8) \\ -\Delta\delta\phi\, EW_o \\ \Delta\delta\phi(NS_0+2A_0/\pi) \\ -\Delta\delta\phi\, R_{EW} t_A \\ -\Delta\delta\phi\, R_{NS} t_A \end{bmatrix} \tag{B.21}$$

As long as the data collected is symmetric about the point in the orbit closest to the guide star, unmodeled signals having the forms

$$\begin{aligned} z_u &= e_{b2o}\sin(2\phi_o)\cos(\phi_r+\delta\phi) \\ z_u &= e_{b2o}\sin(2\phi_o)\sin(\phi_r+\delta\phi) \end{aligned} \tag{B.22}$$

produce no unmodeled errors. Note that these unmodeled errors may be reduced by augmenting the data analysis model to include terms having this time dependence.

Scale Factor Variations

Linear Variation in the Scale Factor

A linear variation in the gyroscope scale factor with time produces and unmodeled error with the form

$$z_u = e_{RCg} t_A\left(\frac{t}{t_A}\right)\left\{\begin{array}{c}\left(NS_0 + R_{NS}t_A\frac{t}{t_A} + A_o\cos\phi_o\right)\cos(\phi_r+\delta\phi) + \\ +\left(EW_0 + R_{EW}t_A\frac{t}{t_A}\right)\sin(\phi_r+\delta\phi)\end{array}\right\} \tag{B.23}$$

where e_{RCg} is the magnitude of the scale factor variation in the period t_A, which is usually taken to be one year. In a one year period, assuming data is collected for half of each orbit, the errors in the second step states are

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} t_A \\ \Delta R_{EW} t_A \end{bmatrix} = \frac{e_{RCg} t_A}{C_g} \begin{bmatrix} 0 \\ 0 \\ R_{NS} t_A / 12 \\ R_{EW} t_A / 12 \\ NS_0 + 2A_o / \pi \\ EW_0 \end{bmatrix} \qquad \text{(B.24)}$$

These results indicate that the unmodeled errors in the gyroscope drift rates due to a linear variation in the scale factor may be reduced by decreasing the average misalignment. The quantity, $NS_0 + 2A_0/\pi$, is the average misalignment in the north-south direction during guide star valid, while EW_0 is the average misalignment in the east-west direction. The present GP-B requirements call for an average misalignment of less than 1 arc second. With this misalignment, a linear drift in the scale factor of 10^{-5} of the nominal value in one year will produce an error in the measured drift rate of 0.01 mas in one year. These results are modified slightly in the case of the long term analysis, but the conclusion is the same – decreasing the average misalignment decreases the unmodeled error in the gyroscope drift rate due to a linear drift in the scale factor. In this case, the unmodeled error may also be reduced by including additional terms in the data analysis model to account for the linear drift in the scale factor.

Scale Factor Variations at the Orbital Frequency

A scale factor variation at the orbital frequency will produce the following unmodeled error in the combined gyroscope and telescope outputs

$$z_u = e_{Cgo} \cos\phi_o \left\{ \begin{array}{l} \left(NS_0 + R_{NS} t_A \dfrac{t}{t_A} + A_o \cos\phi_o \right) \cos(\phi_r + \delta\phi) + \\ \qquad + \left(EW_0 + R_{EW} t_A \dfrac{t}{t_A} \right) \sin(\phi_r + \delta\phi) \end{array} \right\} \qquad \text{(B.25)}$$

From the form of this unmodeled error, it might be expected that this type of error would produce errors in the scale factor and roll phase offset which are proportional to the misalignment. Then, these scale factor and roll phase offset errors, in turn, would produce errors in the estimated misalignments and drift rates. In fact, the analytical expression for the unmodeled errors in the second step states show that this outcome is exactly as expected. The errors in the second step states are

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} t_A \\ \Delta R_{EW} t_A \end{bmatrix} = \begin{bmatrix} e_{Cgo}(NS_o + 2\pi A_o / (3(\pi^2 - 8))) / A_o \\ -e_{Cgo} EW_0 / (C_g A_0) \\ -(\Delta C_g / C_g) NS_o - (\Delta\delta\phi) EW_0 + (e_{Cgo} / C_g) A_0 ((3\pi^2 - 32) / (6(\pi^2 - 8))) \\ -(\Delta C_g / C_g) EW_0 + (\Delta\delta\phi) NS_0 \\ -(\Delta C_g / C_g) R_{NS} t_A - (\Delta\delta\phi) R_{EW} t_A + (e_{Cgo} / C_g)(2 R_{NS} t_A / \pi) \\ -(\Delta C / C_g) R_{EW} t_A + (\Delta\delta\phi) R_{EW} t_A + (e_{Cgo} / C_g)(2 R_{EW} t_A / \pi) \end{bmatrix}$$

(B.26)

Variations in the Roll Phase Offset

The roll phase offset is the difference between the roll phase as measured by the star trackers on the each of the attitude platforms and the normal to the pickup loop for each of the four gyroscopes. In the basic data analysis model, this roll phase offset is considered to be constant with time. The accuracy with which this constant roll phase offset may be determined as a function of time has been discussed in [4]. Variations in this roll phase offset may arise from thermally induced mechanical variations in this angle, errors in the measurement star trackers of the roll phase, of variations in the phase shift of the telescope or gyroscope readout signal at the satellite roll frequency. Early studies [5] pointed out the importance of the stability of this roll phase offset but modeled the variation in the roll phase offset as a random walk. Since the variation is much more likely to be thermally driven, the cases considered here are a linear variation in the roll phase offset and a variation at the orbital frequency.

Linear Variation in the Roll Phase Offset

A small variation in the roll phase offset which is linear in time will produce an unmodeled error in the combined gyroscope and telescope signal having the form

$$z_u = e_{R\phi} t_A \frac{t}{t_A} C_g \left[\left(NS + R_{NS} t_A \frac{t}{t_A} + A_0 \cos\phi_o \right)(-\sin(\phi_r + \delta\phi)) + \left(EW_0 + R_{Ew} t_A \frac{t}{t_A} \right) \cos(\phi_r + \delta\phi) \right] \quad \text{(B.27)}$$

This unmodeled error produces the following errors in the second-step states:

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW} \end{bmatrix} = \frac{e_{RCg}}{C_g} \begin{bmatrix} 0 \\ 0 \\ R_{NS}/12 \\ R_{EW}/12 \\ NS_0 + 2A_o/\pi \\ EW_0 \end{bmatrix} \quad \text{(B.28)}$$

Variations in the Roll Phase Offset at the Orbital Frequency

A small variation in the roll phase offset at the orbital frequency will produce an unmodeled error having the form

$$z_u = e_{\phi o} \cos\phi_o C_g \left[\left(NS_0 + R_{NS} t_A \frac{t}{t_A} + A_0 \cos\phi_o \right)(-\sin(\phi_r + \delta\phi)) + \left(EW_0 + R_{Ew} t_A \frac{t}{t_A} \right) \cos(\phi_r + \delta\phi) \right] \quad \text{(B.29)}$$

which will produce the following unmodeled errors in the second step states:

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS}t_A \\ \Delta R_{EW}t_A \end{bmatrix} = \begin{bmatrix} e_{\phi o} C_g EW_0 / A_0 \\ e_{\phi o} \{NS_o / A_o + 2\pi / [3(\pi^2 - 8)]\} \\ -(\Delta C_g / C_g) NS_o - (\Delta\delta\phi) EW_0 \\ -(\Delta C_g / C_g) EW_0 + (\Delta\delta\phi) NS_0 + e_{\phi_o} A_o (32 - 3\pi^2)/(6(\pi^2 - 8)) \\ -(\Delta C_g / C_g) R_{NS} t_A - (\Delta\delta\phi) R_{EW} t_A + e_{\phi o} 2 R_{EW} t_A / \pi \\ -(\Delta C / C_g) R_{EW} t_A + \Delta\delta\phi R_{NS} t_A - e_{\phi o} 2 R_{NS} t_A / \pi \end{bmatrix} \quad \text{(B.30)}$$

C. Long Term Data Analysis

For the long term data analysis, the basic model is given by equation (A.1) with both the orbital and annual aberration signals included. Approximating the orbital motion of the satellite about the earth and the annual motion of the earth about the sun as circular orbits, this model for the combined gyroscope and telescope signals becomes

$$z = C_g \begin{bmatrix} (NS_0 + R_{NS}(t - t_0) + A_o \cos\phi_o + l_1 \cos\omega_A t + l_2 \sin\omega_A t)\cos(\phi_r + \delta\phi) + \\ + (EW_0 + R_{EW}(t - t_0) + l_3 \cos\omega_A t + l_4 \sin\omega_A t)\sin(\phi_r + \delta\phi) \end{bmatrix} \quad \text{(C.1)}$$

Here, l_1, l_2, l_3, and l_4 are the components of the annual aberration signal, which depend on the right ascension and declination of the guide star, the time at the midpoint of the data collection, and the inclination of the earth's orbit relative to the ecliptic. This equation assumes a circular orbit and is, of course, an approximation, but it is sufficient for estimating the unmodeled errors. With this model the problem may again be broken up into a linear first-step part and a nonlinear second step. The 1 × 10 measurement matrix is

$$H = \Big[\cos\phi_r, \frac{t}{t_A}\cos\phi_r, \cos\phi_A \cos\phi_r, \sin\phi_A \cos\phi_t, \cos\phi_o \cos\phi_r$$
$$sin\phi_r, \frac{t}{t_A}\sin\phi_r, \cos\phi_A \sin\phi_r, \sin\phi_A \sin\phi_t, \cos\phi_o \sin\phi_r \Big] \quad \text{(C.2)}$$

and the corresponding first step states may be written in terms of the second step states as

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \end{bmatrix} = \begin{bmatrix} C_g (NS_0 \cos\delta\phi + EW_0 \sin\delta\phi) \\ C_g (R_{NS} t_A \cos\delta\phi + R_{EW} t_A \sin\delta\phi) \\ C_g (l_1 \cos\delta\phi + l_3 \sin\delta\phi) \\ C_g (l_2 \cos\delta\phi + l_4 \sin\delta\phi) \\ C_g A_0 \cos\delta\phi \\ C_g (-NS_0 \sin\delta\phi + EW_0 \cos\delta\phi) \\ C_g (-R_{NS} t_A \sin\delta\phi + R_{EW} t_A \cos\delta\phi) \\ C_g (-l_1 \sin\delta\phi + l_3 \cos\delta\phi) \\ C_g (-l_2 \sin\delta\phi + l_4 \cos\delta\phi) \\ -C_g A_0 \sin\delta\phi \end{bmatrix} \quad \text{(C.3)}$$

The unmodeled errors for any assumed error in the combined gyroscope and telescope measurement may be calculated here in the same manner as was done for the short term data analysis. Measurement errors will produce errors in the 10 first step states and these errors in turn will produce errors in the 6 second step states. One significant difference between this problem and the short term data analysis problem is that the second step states are over determined. The least squares solution is a weighted average of the scale factor and roll phase offset determined from the various components of the aberration signals. Over short time intervals, these quantities are almost entirely determined from the orbital aberration, while over longer time intervals the annual aberration signal predominates. The appropriate weighting coefficients may be found for any given duration, fraction of the orbit over which the data is collected, and start date. Assuming that the duration is one year and that the data is collected over half of each orbit, the weighting coefficients were calculated with a Matlab symbolic math program. Under these conditions, the errors in the scale factor and roll phase offset are

$$\Delta C_g = \frac{1}{D}\begin{bmatrix} (\pi^2-8)A_o(\Delta y_5\cos\delta\phi - \Delta y_{10}\sin\delta\phi) + \\ +\pi^2\left[(l_1\cos\delta\phi + l_3\sin\delta\phi)\Delta y_3 - (l_1\sin\delta\phi - l_3\cos\delta\phi)\Delta y_8\right] + \\ +(\pi^2-6)\left[(l_2\cos\delta\phi + l_4\sin\delta\phi)\Delta y_4 - (l_2\sin\delta\phi - l_4\cos\delta\phi)\Delta y_9\right] \end{bmatrix}$$

$$\Delta\delta\phi = \frac{1}{D}\begin{bmatrix} (\pi^2-8)A_o(-\Delta y_5\sin\delta\phi - \Delta y_{10}\cos\delta\phi) + \\ +\pi^2\left[(-l_1\sin\delta\phi + l_3\cos\delta\phi)\Delta y_3 - (l_1\cos\delta\phi + l_3\sin\delta\phi)\Delta y_8\right] + \\ +(\pi^2-6)\left[(-l_2\sin\delta\phi + l_4\cos\delta\phi)\Delta y_4 - (l_2\cos\delta\phi + l_4\sin\delta\phi)\Delta y_9\right] \end{bmatrix} \quad \text{(C.4)}$$

where

$$D = (\pi^2-8)A_o^2 + \pi^2(l_1^2 + l_3^2) + (\pi^2-6)(l_2^2 + l_4^2) \quad \text{(C.5)}$$

These equations clearly show the relative weighting of the various components of the aberration signal in determining the errors in the scale factor and roll phase offset.

While the errors in the scale factor and roll phase offset are entirely determined from the y_3, y_4, y_5, and y_8, y_9, and y_{10}, the errors in the average misalignment and drift rate are entirely determined from the other four first step states. In this case, the same equations apply as those for the short term analysis:

$$\begin{bmatrix} \Delta NS \\ \Delta EW \\ \Delta R_{NS}t_A \\ \Delta R_{EW}t_A \end{bmatrix} = \begin{bmatrix} -(\Delta C_g/C_g)NS - (\delta\phi)EW + (\Delta y_1/C_g)\cos\delta\phi - (\Delta y_6/C_g)\sin\delta\phi \\ -(\Delta C_g/C_g)EW + (\delta\phi)NS + (\Delta y_1/C_g)\sin\delta\phi + (\Delta y_6/C_g)\cos\delta\phi \\ -(\Delta C_g/C_g)R_{NS}t_A - (\delta\phi)R_{EW}t_A + (\Delta y_2/C_g)\cos\delta\phi - (\Delta y_7/C_g)\sin\delta\phi \\ -(\Delta C_g/C_g)R_{EW}t_A + (\delta\phi)R_{NS}t_A + (\Delta y_2/C_g)\sin\delta\phi + (\Delta y_7/C_g)\cos\delta\phi \end{bmatrix} \quad \text{(C.6)}$$

From these sets of equations the errors in the second step states may be calculated from the errors in the first step states.

Bias Variations

Bias Variations at the Satellite Roll Frequency

Bias Variations which have a constant amplitude and phase are indistinguishable from an average misalignment of the gyroscope spin axis. In this case for the long-term data analysis, the unmodeled error is exactly the same as that for the short-term data analysis. The unmodeled signal corresponding to an average misalignment in the NS direction is given by equation (B.5), and the only nonzero unmodeled error is the error in the average misalignment in the north-south direction, $\Delta NS = e_{NS}$, as shown in equation (B.6). Similarly, the unmodeled signal corresponding to an average misalignment in the EW direction is given by equation (B.7), and the only nonzero unmodeled error is the error in the average misalignment in the east-west direction, $\Delta NS = e_{NS}$, as shown in equation (B.8). It would be surprising if the result was anything other than this expected result.

Roll Frequency Bias Variations Which Vary Linearly with Time

Roll Frequency Bias Variations where the amplitude varies linearly with time have exactly the same time signature as a gyroscope drift rate. The unmodeled errors are exactly the same as in the case of the short term data analysis. The unmodeled components of the combined gyroscope and telescope signal are given by equations (B.9) and (B.11), and the only nonzero unmodeled errors are those for the gyroscope drift rate in the north-south and east-west directions as shown in equations (B.10) and (B.12).

Roll Frequency Bias Variations Modulated at the Orbital Frequency

For a roll frequency bias variation which is modulated at the orbital frequency, where the phase of the roll frequency variation corresponds to the expected phase of the orbital aberration signal, the unmodeled signal is

$$z_u = e_{NSO} \cos\phi_o \cos(\phi_r + \delta\phi) \tag{C.7}$$

and the corresponding unmodeled errors in the second step states are (assuming a 1 year duration)

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW} \end{bmatrix} = \begin{bmatrix} (e_{NSO} / A_o)C_g f \\ 0 \\ -(\Delta C_g / C_g)(NS + 2A_o / \pi) + 2e_{NSO} / \pi \\ -(\Delta C_g / C_g)EW_0 \\ -(\Delta C_g / C_g)(R_{NS} + 6l_2 / \pi) \\ -(\Delta C_g / C_g)(R_{EW} + 6l_4 / \pi) \end{bmatrix} \tag{C.8}$$

where f is the weighting function for the orbital aberration in determining the scale factor

$$f = \frac{A_o^2(\pi^2 - 8)}{A_o^2(\pi^2 - 8) + \pi^2(l_1^2 + l_3^2) + (\pi^2 - 6)(l_2^2 + l_4^2)} \quad \text{(C.9)}$$

Since the orbital aberration signal is considerably smaller than the annual aberration signal, the relative error in the scale factor is significantly smaller than the relative error in the orbital aberration signal. Therefore, for the long term analysis these changes are not significant.

For a roll frequency bias variation which is modulated at the orbital frequency, where the phase of the roll frequency variation is out of phase with the orbital aberration signal, the unmodeled signal is

$$z_u = e_{EWO} \cos\phi_o \sin(\phi_r + \delta\phi) \quad \text{(C.10)}$$

and the corresponding unmodeled errors in the second step states are (assuming a 1 year duration)

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW} \end{bmatrix} = \begin{bmatrix} 0 \\ -(e_{EWO} / A_0)f \\ -\Delta\delta\phi\,(EW_o) \\ \Delta\delta\phi\,(NS_0 + 2A_0/\pi) - 2e_{EWO}/\pi \\ -\Delta\delta\phi(R_{NS} + 6l_4/\pi) \\ \Delta\delta\phi(R_{EW} + 6l_2/\pi) \end{bmatrix} \quad \text{(C.11)}$$

where f is the same weighting function as given above. Since f is considerably smaller than one, the effects of a bias variation at the orbital frequency would be considerable reduced for data taken over a one year period. However, if the scale factor and roll phase offset were determined from the orbital aberration and annual aberration data separately, then these results would not agree to within their standard deviations.

Roll Frequency Bias Variations Modulated at the Annual Frequency

The annual aberration signal produces a shift in the measured position of the gyroscopes relative to the guide star which varies in the both of the inertially fixed directions at an annual frequency. A plot of the annual aberration signal projected onto a plane perpendicular to the direction to the guide star is an ellipse. For roll frequency bias variations modulated at the annual frequency there are four possible cases:

(1) If the roll phase and annual modulation corresponds to the annual aberration signal, then this modulation will produce an error in the scale factor with the corresponding errors in the position and drift rate.

(2) If the roll phase is shifted by 90^0, but the annual modulation corresponds to the annual aberration signal, there will be an error in the roll phase offset with corresponding errors in the position and drift rates of the gyroscopes.

(3) If the roll phase corresponds to the annual modulation, but the annual modulation is shifted by 90^0, then there will be an error in parallax signal. Since parallax signal is assumed to be known for the basic measurement model, there is not net error is the second step states.

(4) For the case where the roll phase and the annual modulation are both shifted by 90^0, there is no net experimental error.

Each of these four cases are treated separately below

If the annual modulation of the roll frequency bias variation has the same time signature as the annual aberration signal then the unmodeled signal has the form

$$\Delta z = e_{bA1}\left[(l_1\cos\phi_a + l_2\sin\phi_a)\cos(\phi_r + \delta\phi) + (l_3\cos\phi_a + l_4\sin\phi_a)\sin(\phi_r + \delta\phi)\right] \quad (C.12)$$

With this unmodeled error the corresponding error in the second step states is

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW} \end{bmatrix} = \begin{bmatrix} e_{bA1}(1-f) \\ 0 \\ -\left(\Delta C_g / C_g\right)\left(NS_o + 2A_o/\pi\right) \\ -\left(\Delta C_g / C_g\right)\left(EW_0\right) \\ -\Delta C_g / C_g\left(R_{NS}\right) + (e_{bA1}/C_g) f\left(R_{NS} + 6l_2/\pi\right) \\ -\Delta C_g / C_g\left(R_{EW}\right) + (e_{bA1}/C_g) f\left(R_{EWS} + 6l_4/\pi\right) \end{bmatrix} \quad (C.13)$$

Here, f is the weighting factor for the orbital aberration signal defined above. For a data set of one year or more, this function f is much less than one. In this case, the error in the scale factor is approximately equal to e_{bA1}, and the fractional error in the drift rates is approximately equal to fractional error in the scale factor.

If the roll phase of the unmodeled signal is shifted by 90^0, the unmodeled signal becomes

$$\Delta z = e_{bA2}\left[-(l_1\cos\phi_a + l_2\sin\phi_a)\sin(\phi_r + \delta\phi) + (l_3\cos\phi_a + l_4\sin\phi_a)\cos(\phi_r + \delta\phi)\right] \quad (C.14)$$

the corresponding error in the second step states is

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW} \end{bmatrix} = \begin{bmatrix} 0 \\ e_{bA2}(1-f) \\ -\left(\Delta\delta\phi\right)\left(EW_o\right) \\ +\left(\Delta\delta\phi\right)\left(NS_o + 2A_o/\pi\right) \\ -\left(\Delta\delta\phi\right)\left(R_{EW}\right) + (e_{bA1}/C_g) f\left(R_{EW} + 6l_4/\pi\right) \\ +\left(\Delta\delta\phi\right)\left(R_{NS}\right) - (e_{bA1}/C_g) f\left(R_{NS} + 6l_2/\pi\right) \end{bmatrix} \quad (C.15)$$

In this case the annual modulation of the roll frequency variation in the bias produces an error in the roll phase offset but no error in the scale factor.

The other two errors in the annual modulation of the roll frequency bias variation occur if the phase of the annual modulation is shifted by 90^0. If the unmodeled signal lies in the same plane as the annual aberration, it is

$$\Delta z = e_{bA3}\left[(-l_1\sin\phi_a + l_2\cos\phi_a)\cos(\phi_r + \delta\phi) + (-l_3\sin\phi_a + l_4\cos\phi_a)\sin(\phi_r + \delta\phi)\right].$$

In this case the errors in the second step states are

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW} \end{bmatrix} = \begin{bmatrix} 6e_{bA3}(l_1l_2 + l_3l_4)/D \\ e_{bA3}2(\pi^2 - 3)(l_1l_4 - l_2l_3)/(C_gD) \\ -(\Delta C_g/C_g)(NS_o + 2A_o/\pi) - (\Delta\delta\phi)(EW_o) \\ -(\Delta C_g/C_g)(EW_o) + (\Delta\delta\phi)(NS_o + 2A_o/\pi) \\ -(\Delta C_g/C_g)(R_{NS} + 6l_2/\pi) - (\Delta\delta\phi)(R_{EW} + 6l_4/\pi) + 6l_1(e_{bA3}/C_g)/\pi \\ -(\Delta C_g/C_g)(R_{EW} + 6l_4/\pi) + (\Delta\delta\phi)(R_{NS} + 6l_2/\pi) + 6l_3(e_{bA3}/C_g)/\pi \end{bmatrix} \quad (C.16)$$

where

$$D = \pi^2(l_1^2 + l_3^2) + (\pi^2 - 6)(l_2^2 + l_4^2) + (\pi^2 - 8)A_0^2)$$

If the unmodeled signal is 90^0 out of phase with the annual aberration but in a plane perpendicular to the annual aberration then the unmodeled signal is

$$\Delta z = e_{bA4}\left[-(-l_1\sin\phi_a + l_2\cos\phi_a)\sin(\phi_r + \delta\phi) + (-l_3\sin\phi_a + l_4\cos\phi_a)\cos(\phi_r + \delta\phi)\right] \quad (C.17)$$

and the errors in the second step states are

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW} \end{bmatrix} = \begin{bmatrix} e_{bA4}2(\pi^2 - 3)(l_1l_4 - l_2l_3)/D \\ 6e_{bA4}(l_1l_2 + l_3l_4)/(C_gD) \\ -(\Delta C_g/C_g)(NS + 2A_0/\pi) - (\Delta\delta\phi)EW \\ -(\Delta C_g/C_g)EW + (\Delta\delta\phi)(NS + 2A_0/\pi) \\ -(\Delta C_g/C_g)(R_{NS} + 6l_2/\pi) - (\Delta\delta\phi)(R_{EW} + 6l_4/\pi) + 6l_3(e_{bA4}/C_g)/\pi \\ -(\Delta C_g/C_g)(R_{EW} + 6l_4/\pi) + (\Delta\delta\phi)(R_{NS} + 6l_2/\pi) - 6l_1(e_{bA4}/C_g)/\pi \end{bmatrix} \quad (C.18)$$

Scale Factor Variations

Modulation of the Scale Factor at the Orbital Frequency

If the scale factor in modulated at the orbital frequency, the unmodeled signal is

$$z_u = e_{OCg}\cos\phi_o\begin{bmatrix}(NS + R_{NS}t + l_1\cos\phi_a + l_2\sin\phi_a + A_o\cos\phi_o)\cos(\phi_r + \delta\phi) + \\ + (EW + R_{EW}t + l_3\cos\phi_a + l_4\sin\phi_a)\sin(\phi_r + \delta\phi)\end{bmatrix} \quad (C.19)$$

This unmodeled errors in the second step states are

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW} \end{bmatrix} = \begin{bmatrix} e_{OCg}\left[2/\pi + (A_oNS(\pi^2 - 16) + (4/3\pi)A_o^2(12 - \pi^2))/D\right] \\ -e_{OCg}A_oEW(\pi^2 - 16)/(C_gD) \\ -(\Delta C_g/C_g)(NS + 2A_0/\pi) - (\Delta\delta\phi)EW + (e_{OCg}/C_g)(4NS/\pi + A_o/2) \\ -(\Delta C_g/C_g)EW + (\Delta\delta\phi)(NS + 2A_0/\pi) + (e_{OCg}/C_g)(4EW/\pi) \\ -(\Delta C_g/C_g)(R_{NS} + 6l_2/\pi) - (\Delta\delta\phi)(R_{EW} + 6l_4/\pi) + 2(e_{OCg}/C_g)(R_{NS} + 6l_2/\pi)/\pi \\ -(\Delta C_g/C_g)(R_{EW} + 6l_4/\pi) + (\Delta\delta\phi)(R_{NS} + 6l_2/\pi) + 2(e_{OCg}/C_g)(R_{EW} + 6l_4/\pi)/\pi \end{bmatrix}$$

(C.20)

Modulation of the Scale Factor at the Annual Frequency

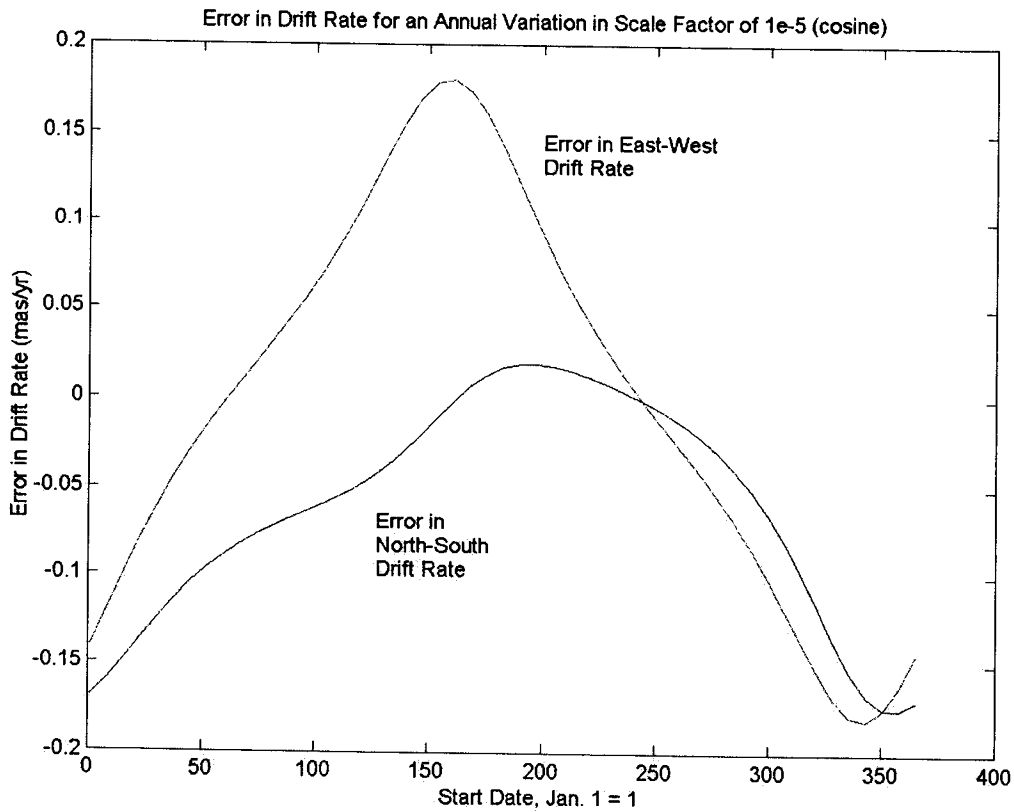
If the scale factor is modulated as the cosine of the annual frequency, the unmodeled signal is

$$z_u = e_{CgA}\cos\phi_a \begin{bmatrix} (NS + R_{NS}t + l_1\cos\phi_a + l_2\sin\phi_a + A_o\cos\phi_o)\cos(\phi_r + \delta\phi) + \\ +(EW + R_{EW}t + l_3\cos\phi_a + l_4\sin\phi_a)\sin(\phi_r + \delta\phi) \end{bmatrix} \quad \text{(C.21)}$$

and the corresponding errors in the second step states are

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW} \end{bmatrix} = \begin{bmatrix} e_{CgA}(\pi^2(l_1(NS + (2/\pi)A_o) + l_3 EW) + (6/\pi - \pi/4)(l_2 R_{NS} + l_4 R_{EW}) + (3/2)(l_2^2 + l_4^2))/D \\ e_{CgA}(\pi^2(l_3(NS + (2/\pi)A_o) - l_1 EW) - (6/\pi - \pi/4)(l_2 R_{EW} - l_4 R_{NS}))/(C_g D) \\ -(\Delta C_g / C_g)(NS + 2A_0/\pi) - (\Delta\delta\phi)EW + (e_{CgA}/C_g)l_1/2 \\ -(\Delta C_g / C_g)EW + (\Delta\delta\phi)(NS + 2A_0/\pi) + (e_{CgA}/C_g)l_3/2 \\ -(\Delta C_g / C_g)(R_{NS} + 6l_2/\pi) - (\Delta\delta\phi)(R_{EW} + 6l_4/\pi) - 6(e_{CgA}/C_g)(R_{NS} + \pi l_2/4)/\pi^2 \\ -(\Delta C_g / C_g)(R_{EW} + 6l_4/\pi) + (\Delta\delta\phi)(R_{NS} + 6l_2/\pi) - 6(e_{CgA}/C_g)(R_{EW} + \pi l_4/4)/\pi^2 \end{bmatrix} \quad \text{(C.22)}$$

These unmodeled errors in the gyroscope drift rates are shown in the figure below:



If the scale factor is modulated as the sine of the annual frequency, the unmodeled signal is

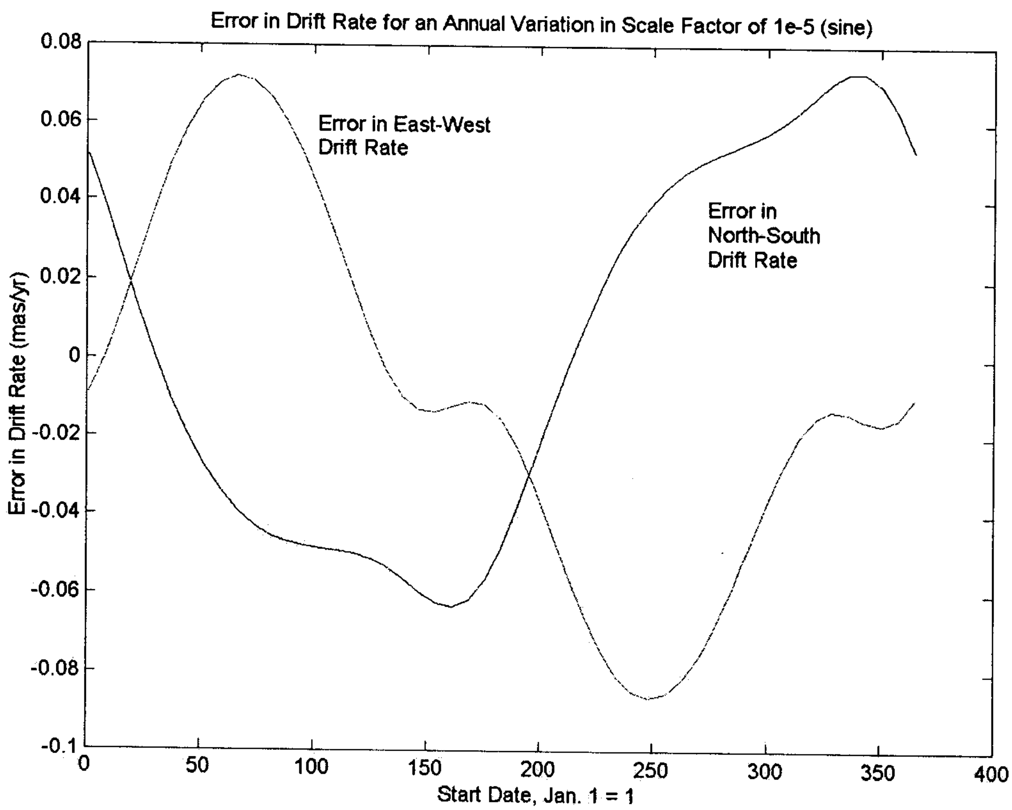
$$z_u = e_{CgA}\sin\phi_a\begin{bmatrix}(NS + R_{NS}t + l_1\cos\phi_a + l_2\sin\phi_a + A_o\cos\phi_o)\cos(\phi_r + \delta\phi) + \\ +(EW + R_{EW}t + l_3\cos\phi_a + l_4\sin\phi_a)\sin(\phi_r + \delta\phi)\end{bmatrix} \qquad (C.23)$$

and the corresponding error in the second step states are

$$\begin{bmatrix}\Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW}\end{bmatrix} = \begin{bmatrix} e_{CgA}((\pi^2-6)(l_2(NS+(2/\pi)A_o)+l_4EW)-\pi/4(l_1R_{NS}+l_3R_{EW})+(3/2)(l_1l_2+l_3l_4))/D \\ e_{CgA}((\pi^2-6)(l_4(NS+(2/\pi)A_o)-l_2EW)+\pi/4(l_1R_{EW}-l_3R_{NS})+(3/2)(l_1l_4-l_2l_3))/(C_gD) \\ -(\Delta C_g/C_g)(NS+2A_0/\pi)-(\Delta\delta\phi)EW+(e_{CgA}/C_g)(R_{NS}/(2\pi)+l_2/2) \\ -(\Delta C_g/C_g)EW+(\Delta\delta\phi)(NS+2A_0/\pi)+(e_{CgA}/C_g)(R_{EW}/(2\pi)+l_4/2) \\ -(\Delta C_g/C_g)(R_{NS}+6l_2/\pi)-(\Delta\delta\phi)(R_{EW}+6l_4/\pi)+6(e_{CgA}/C_g)(NS+2A_o/\pi+l_1/4)/\pi \\ -(\Delta C_g/C_g)(R_{EW}+6l_4/\pi)+(\Delta\delta\phi)(R_{NS}+6l_2/\pi)+6(e_{CgA}/C_g)(EW+l_3/4)/\pi \end{bmatrix}$$

(C.24)

The unmodeled errors in the gyroscope drift rates are shown in the figure below



Linear Drift in Scale Factor

For the case of a linear drift in the scale factor, the unmodeled signal is given by

$$z_u = e_{RCg}\left(\frac{t}{t_a}\right)\begin{bmatrix}(NS + R_{NS}t + l_1\cos\phi_a + l_2\sin\phi_a + A_o\cos\phi_o)\cos(\phi_r + \delta\phi) + \\ +(EW + R_{EW}t + l_3\cos\phi_a + l_4\sin\phi_a)\sin(\phi_r + \delta\phi)\end{bmatrix} \quad \text{(C.25)}$$

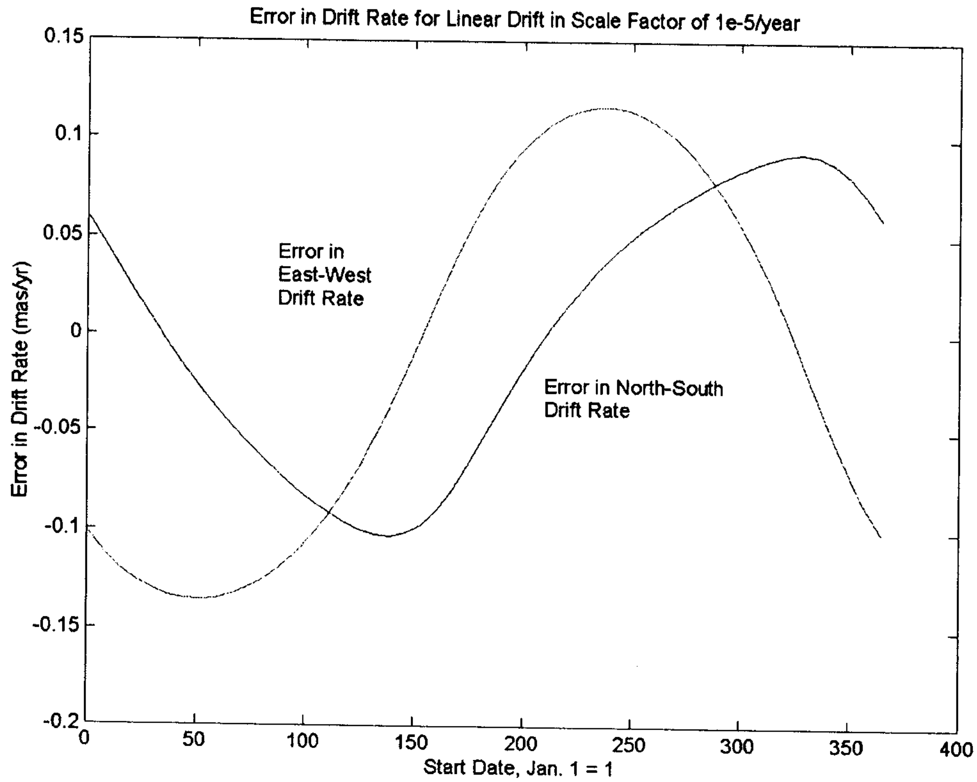
Two different Matlab symbolic math programs were used to calculate the unmodeled errors due to this unmodeled signal. The first program calculated the quantity $H^T z_u$, which involves finding the integrals over the annual period. This calculation resulted in numberous terms in harmonics of the orbital period. These terms were negected and only the d.c. terms were used in a second Matlab symbolic math program. Using this program the unmodeled errors in the second step states were found to be. The results are

$$\begin{bmatrix}\Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS}t_A \\ \Delta R_{EW}t_A\end{bmatrix} = \begin{bmatrix} -e_{RCg}\left[R_{EW}t_A l_3 + R_{NS}t_A l_1 + (\pi/2 - 6/\pi)(l_1 l_2 + l_3 l_4)\right]/D \\ (e_{RCg}/C_g)\left[R_{EW}t_A l_1 - R_{NS}t_A l_3 + (6/\pi)(l_1 l_4 - l_2 l_3)\right]/D \\ -(\Delta C_g/C_g)(NS + 2A_0/\pi) - (\Delta\delta\phi)EW + (e_{RCg}/C_g)(R_{NS}t_A + 6l_2/\pi)/12 \\ -(\Delta C_g/C_g)EW + (\Delta\delta\phi)(NS + 2A_0/\pi) + (e_{RCg}/C_g)(R_{EW}t_A + 6l_4/\pi)/12 \\ -(\Delta C_g/C_g)(R_{NS}t_A + 6l_2/\pi) - (\Delta\delta\phi)(R_{EW}t_A + 6l_4/\pi) + (e_{RCg}/C_g)(NS + 2A_0/\pi - 6l_1/\pi^2) \\ -(\Delta C_g/C_g)(R_{EW}t_A + 6l_4/\pi) + (\Delta\delta\phi)(R_{NS}t_A + 6l_2/\pi) + (e_{RCg}/C_g)(EW - 6l_3/\pi^2) \end{bmatrix} \quad \text{(C.26)}$$

where

$$D = \pi^2\left(l_1^2 + l_2^2 + l_3^2 + l_4^2\right) - 6\left(l_2^2 + l_4^2\right) + \left(\pi^2 - 8\right)A_0$$

The unmodeled error due to the linear drift in the scale factor is shown in the figure below:



These results agree with the results for the short term data analysis if the components of the annual aberration signal, l_i, are set equal to zero. In that case there was no unmodeled error in the scale factor or the roll phase offset. However, here there is a significant unmodeled error in the scale factor and the roll phase offset. These errors, in turn, contribute to the unmodeled errors in the average misalignment and the gyroscope drift rate. The dominant contribution to the unmodeled error in the gyroscope drift rate is the last term in each expression. A plot of the unmodeled error various start dates is shown in the figure below. The magnitude of this last term, and hence the unmodeled error in the drift rate may be significantly reduced by adjusting the initial misalignment.

Effects of Temporal Variation in the Roll Phase Offset

The roll phase offset is azimuthal angle between the star tracker and the normal to each of the four pickup loops around the gyroscopes. Errors in the determination of this angle will shift the orientation of the measured drift rate so that some of the drift rate in the north-south direction will appear in the east-west direction and vice versa. To meet the top level GP-B requirement of 0.5 mas/year this angle must be known to an accuracy of better than 7.5×10^{-5} radians (15 arc seconds). This angle may be determined from the orbital and annual aberration signals because the direction of these effects are known from the measured velocity and position of the spacecraft.

Early covariance analyses [5] identified temporal variations in this angle as an important contribution to the overall experiment error. In this study the temporal variation was assumed to be a random walk and a Kalman filter was used to model this temporal variation. In fact, variations in this angle are likely to be thermally driven and have significant components at the orbital and annual frequencies. Here, the effects of temporal variations in this angle on the parameters determined from the basic data analysis are calculated. Note that these errors are worst case numbers since they assume that the basic data analysis model is not augmented to include these effects. The analysis of data from the GP-B satellite will, of course, investigate the possibility of these effects.

Phase Shift Variations at the Satellite Roll Frequency

Variations in the roll phase offset at the satellite roll frequency unmodeled signals having the form

$$z_u = e_{\phi r}\begin{Bmatrix}\cos\phi_r \\ \sin\phi_r\end{Bmatrix} C_g \begin{bmatrix} -(NS_o + R_{NS}(t-t_0) + a_{NS})\sin(\phi_r + \delta\phi) + \\ +(\mathrm{EW}_0 + R_{EW}(t-t_0) + a_{EW})\cos(\phi_r + \delta\phi) \end{bmatrix} \quad \text{(C.27)}$$

The product of the two terms at the satellite roll phase leads to a rectified dc signal and an component at twice the satellite roll rate. Since neither of these frequencies are important to the GP-B data reduction, they produce no significant errors in either the measured misalignments or drift rates.

Phase Shift Variations at the Orbital Frequency

Phase shifts at the satellite orbital frequency may contribute small errors in the long term data analysis. This temporal variation in the angle between the star tracker and the normal to the pickup loop leads to unmodeled signals having the form

$$z_u = e_{\phi o}\begin{Bmatrix}\cos\phi_o \\ \sin\phi_o\end{Bmatrix} C_g \begin{bmatrix} -(NS_o + R_{NS}(t-t_0) + a_{NS})\sin(\phi_r + \delta\phi) + \\ +(\mathrm{EW}_0 + R_{EW}(t-t_0) + a_{EW})\cos(\phi_r + \delta\phi) \end{bmatrix} \quad \text{(C.28)}$$

Here, a_{NS} and a_{EW} include the usual orbital and annual aberration terms. Terms which vary at the sine of the orbital phase produce no significant errors in the second-step states. However, those terms which vary as the cosine of the orbital phase produce the following errors in the second step states:

$$\begin{bmatrix}\Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW}\end{bmatrix} = \begin{bmatrix} e_{\phi o} C_g \left[A_o EW(\pi^2 - 8)\right]/D \\ e_{\phi o}\left[2/\pi + (A_o NS(\pi^2-8) + (4/3\pi)A_o{}^2(6-\pi^2))/D\right] \\ -(\Delta C_g/C_g)(NS + 2A_0/\pi) - (\Delta\delta\phi)EW + 2e_{\phi o}EW/\pi \\ -(\Delta C_g/C_g)EW + (\Delta\delta\phi)(NS + 2A_0/\pi) - 2e_{\phi o}(NS + \pi A_o/4)/\pi \\ -(\Delta C_g/C_g)(R_{NS} + 6l_2/\pi) - (\Delta\delta\phi)(R_{EW} + 6l_4/\pi) + 2e_{\phi o}(R_{EW} + 6l_4/\pi)/\pi \\ -(\Delta C_g/C_g)(R_{EW} + 6l_4/\pi) + (\Delta\delta\phi)(R_{NS} + 6l_2/\pi) - 2e_{\phi o}(R_{NS} + 6l_2/\pi)/\pi \end{bmatrix}$$

(C.29)

Phase Shift Variations at the Annual Frequency

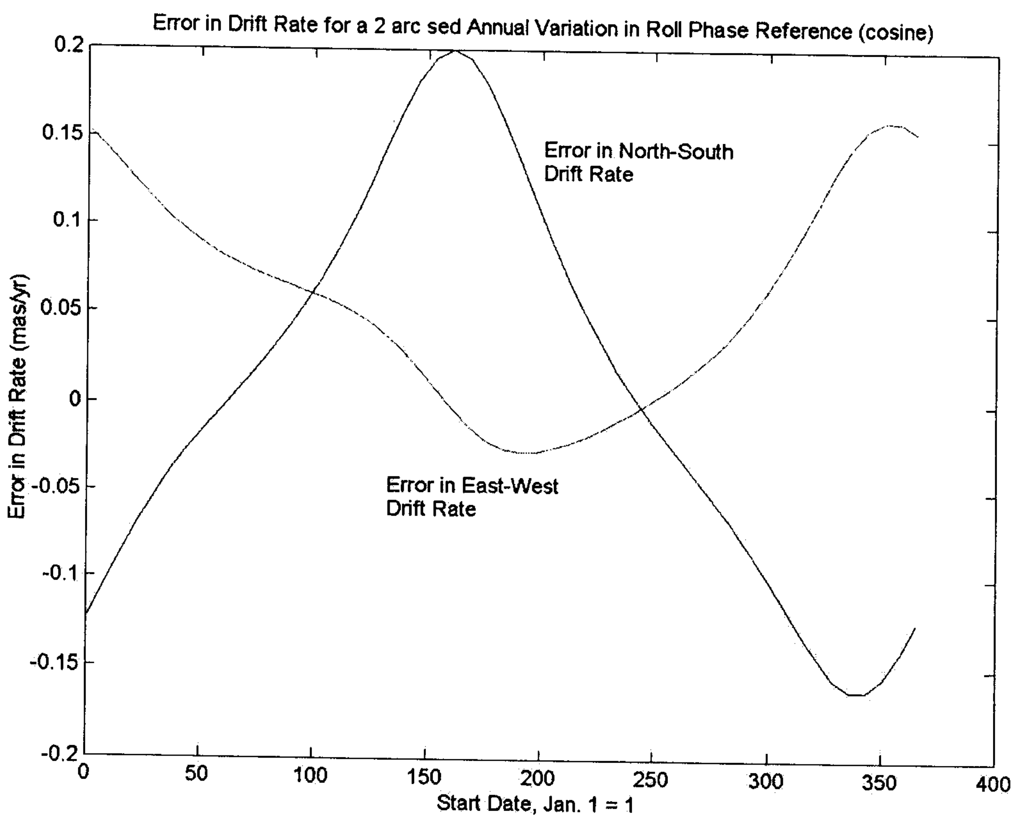
If the roll phase offset varies at an annual frequency, the unmodeled signal is

$$z_u = e_{\phi A}\begin{Bmatrix}\cos\phi_a \\ \sin\phi_a\end{Bmatrix} C_g \begin{bmatrix} -(NS_o + R_{NS}(t-t_0) + a_{NS})\sin(\phi_r + \delta\phi) + \\ \qquad + (\mathrm{EW}_0 + R_{EW}(t-t_0) + a_{EW})\cos(\phi_r + \delta\phi) \end{bmatrix} \tag{C.30}$$

In the first case, where the roll phase offset is modulated as the cosine of the annual frequency, the unmodeled errors in the second step states are

$$\begin{bmatrix}\Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW}\end{bmatrix} = \begin{bmatrix} e_{\phi A} C_g(\pi^2(-l_3(NS + (2/\pi)A_o) + l_1 EW) + (6/\pi - \pi/4)(l_2 R_{EW} - l_4 R_{NS}))/D \\ e_{\phi A}(\pi^2(l_3 EW + l_1(NS + (2/\pi)A_o)) + (6/\pi - \pi/4)(l_2 R_{NS} + l_4 R_{EW}) + (3/2)(l_2^2 + l_4^2))/D \\ -(\Delta C_g / C_g)(NS + 2A_0/\pi) - (\Delta\delta\phi)EW + e_{\phi A} l_3/2 \\ -(\Delta C_g / C_g)EW + (\Delta\delta\phi)(NS + 2A_0/\pi) - e_{\phi A} l_1/2 \\ -(\Delta C_g / C_g)(R_{NS} + 6l_2/\pi) - (\Delta\delta\phi)(R_{EW} + 6l_4/\pi) - 6e_{\phi A}(R_{EW} + \pi l_4/4)/\pi^2 \\ -(\Delta C_g / C_g)(R_{EW} + 6l_4/\pi) + (\Delta\delta\phi)(R_{NS} + 6l_2/\pi) + 6e_{\phi A}(R_{NS} + \pi l_2/4)/\pi^2 \end{bmatrix} \tag{C.31}$$

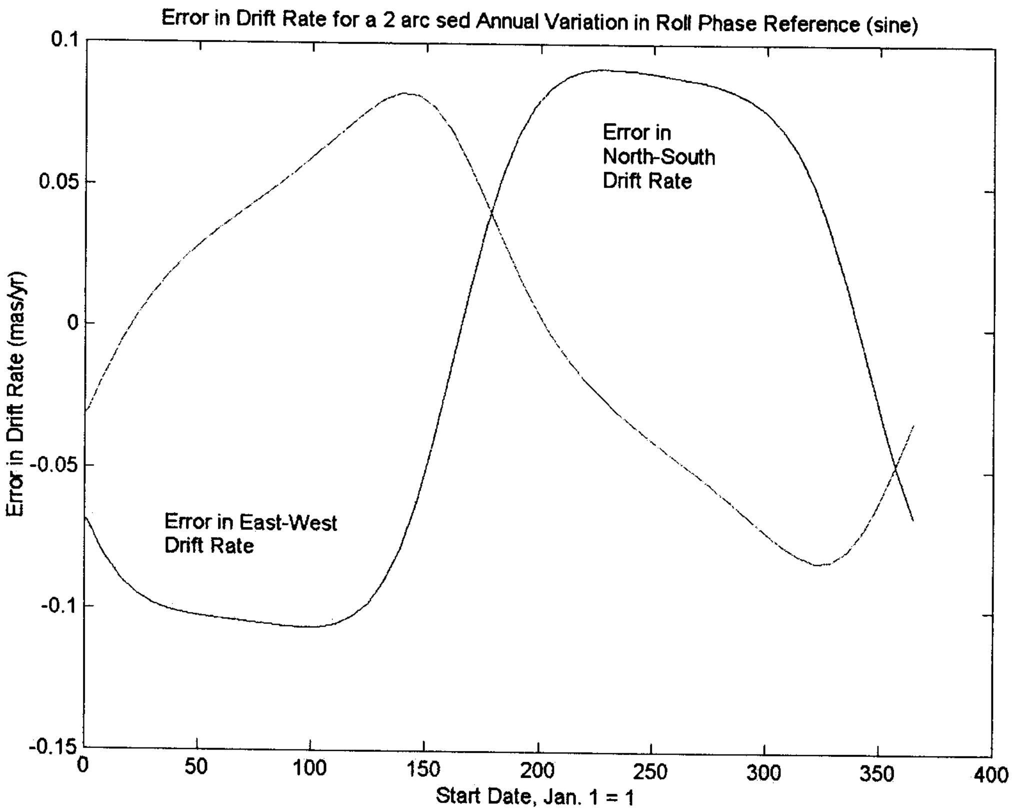
The unmodeled error in the gyroscope drift rate due to a variation in the roll phase reference at he cosine of the annual period is shown in the figure below.



In the second case, where the roll phase offset is modulated as the sine of the annual frequency, the unmodeled errors in the second step states are

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} \\ \Delta R_{EW} \end{bmatrix} = \begin{bmatrix} e_{\phi A} C_g((\pi^2-6)(l_2 EW - l_4(NS+2A_o/\pi)) + \pi/4(l_3 R_{NS} - l_1 R_{EW}) + (3/2)(l_2 l_3 - l_1 l_4))/D \\ e_{\phi A}((\pi^2-6)(l_2(NS+2A_o/\pi) + l_4 EW) - \pi/4(l_1 R_{NS} + l_3 R_{EW}) + (3/2)(l_1 l_2 + l_3 l_4))/D \\ -(\Delta C_g/C_g)(NS+2A_0/\pi) - (\Delta\delta\phi)EW + e_{\phi A}(R_{EW}/\pi + l_2)/2 \\ -(\Delta C_g/C_g)EW + (\Delta\delta\phi)(NS+2A_0/\pi) - e_{\phi A}(R_{NS}/\pi + l_4)/2 \\ -(\Delta C_g/C_g)(R_{NS}+6l_2/\pi) - (\Delta\delta\phi)(R_{EW}+6l_4/\pi) + 6e_{\phi A}(EW - l_3/4)/\pi \\ -(\Delta C_g/C_g)(R_{EW}+6l_4/\pi) + (\Delta\delta\phi)(R_{NS}+6l_2/\pi) - 6e_{\phi A}(NS+2A_o/\pi - l_1/4)/\pi \end{bmatrix} \tag{C.33}$$

These unmodeled errors in the gyroscope drift rates are shown in the figure below



Linear Drift in the Phase Shift

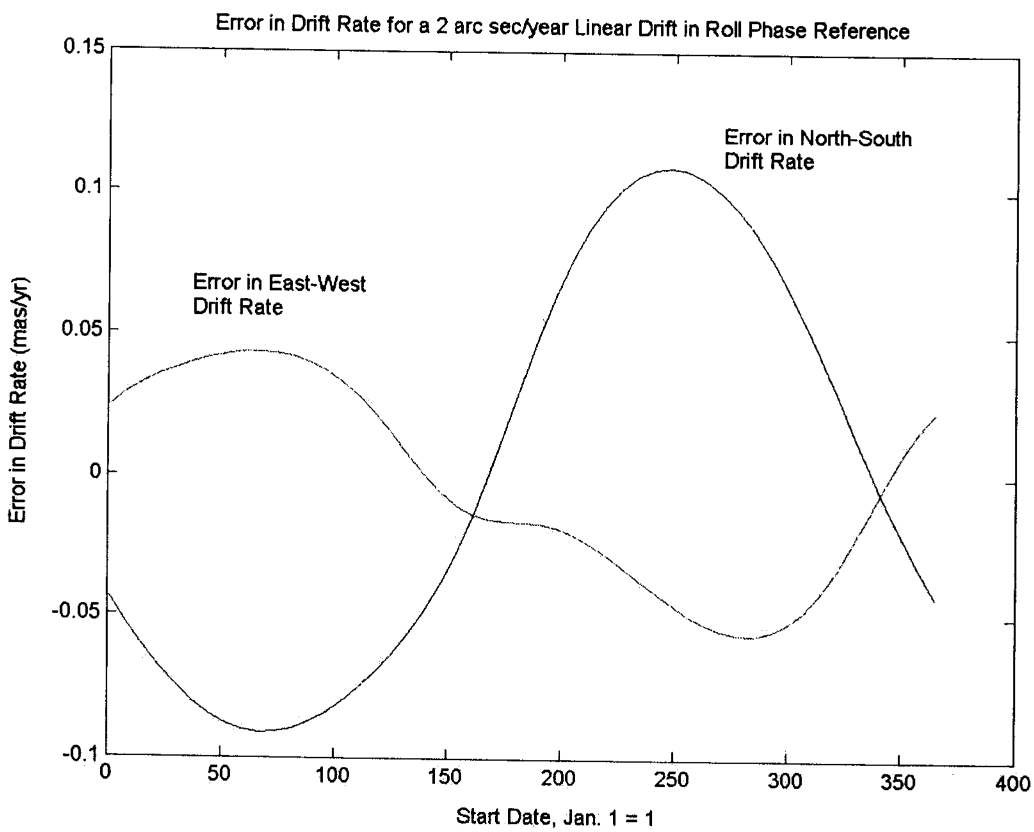
A linear drift in the measured roll phase will produce an unmodeled error in the output of the combined gyroscope and telescope signals given by

$$z_u = e_{\phi R} t C_g \begin{bmatrix} -(NS_o + R_{NS}(t-t_0) + a_{NS})\sin(\phi_r + \delta\phi) + \\ + (\mathrm{EW}_0 + R_{EW}(t-t_0) + a_{EW})\cos(\phi_r + \delta\phi) \end{bmatrix} \quad (C.34)$$

and the corresponding errors in the second step states are

$$\begin{bmatrix} \Delta C_g \\ \Delta\delta\phi \\ \Delta NS \\ \Delta EW \\ \Delta R_{NS} t_A \\ \Delta R_{EW} t_A \end{bmatrix} = \begin{bmatrix} e_{\phi R} C_g \left[-R_{EW} t_A l_1 + R_{NS} t_A l_3 + 6/\pi (l_2 l_3 - l_1 l_4) + (2-\pi) l_4 A_o \right] / D \\ e_{\phi R} \left[-R_{EW} t_A l_3 - R_{NS} t_A l_1 + (6/\pi - \pi/2)(l_1 l_2 + l_3 l_4) - (2-\pi) l_2 A_o \right] / D \\ -(\Delta C_g / C_g)(NS + 2A_0/\pi) - (\Delta\delta\phi) EW + e_{\phi R}(R_{EW} t_A + 6 l_4/\pi)/12 \\ -(\Delta C_g / C_g) EW + (\Delta\delta\phi)(NS + 2A_0/\pi) - e_{\phi R}(R_{NS} t_A + 6 l_2/\pi)/12 \\ -(\Delta C_g / C_g)(R_{NS} t_A + 6 l_2/\pi) - (\Delta\delta\phi)(R_{EW} t_A + 6 l_4/\pi) + e_{\phi R}(EW - 6 l_3/\pi^2) \\ -(\Delta C_g / C_g)(R_{EW} t_A + 6 l_4/\pi) + (\Delta\delta\phi)(R_{NS} t_A + 6 l_2/\pi) - e_{\phi R}(NS + 2A_0/\pi - 6 l_1/\pi^2) \end{bmatrix} \quad (C.35)$$

The calculated unmodeled errors in the gyroscope drift rates are shown in the figure below:



D. Conclusion

These results give explicit expressions for unmodeled errors due to variations in the bias, scale factor, and roll phase offset at the critical frequencies. They may be used to evaluate the unmodeled errors and as a criteria for when the data analysis model needs to be augmented to include additional observable effects.

References

1. Haupt, G.T., *Development and Experimental Verification of a Nonlinear Data Reduction Algorithm for the Gravity Probe B Relativity Mission*, in *Dept. of Aeronautics and Astronautics*. 1996, Stanford University. p. 194.
2. Keiser, G.M., M. Heifetz, and A.S. Silbergleit, *New Error Tree*, Gravity Probe B, Stanford University, S0292, Rev. A,
3. Heifetz, M.I. and G.M. Keiser, *Gravity Probe B Data Reduction: Analysis of Unmodeled Errors*, Gravity Probe B, Stanford University, S0354, October, 1998
4. Keiser, G.M., *Analytical Solution to the Gravity Probe B Covariance Matrix*, Gravity Probe B, Stanford University, S0351, October 21, 1998
5. Vassar, R., *Error Analysis for the Stanford Relativity Gyroscope Experiment*, . 1982, Stanford University.