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Gravity Probe B Relativity Mission

VERIFICATION OF T002 REQUIREMENT 5, Part A: LINEARITY OF GYROSCOPE READOUT SYSTEM

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List of Acronyms, Abbreviations, and Symbols

β_b	London moment equivalent angle of output bias
β_{dc}	D.C. component of London moment equivalent angle
β_c	Amplitude of London moment equivalent angle at calibration signal frequency
β_L	Amplitude of London moment equivalent angle at satellite roll frequency
β_L^{max}	Full scale London moment equivalent angle for a given range and gain of SQUID readout electronics
$\varepsilon_2,\varepsilon_3,\ldots$	Nonlinearity coefficients
$\delta[\ ]$	Change in quantity within brackets
δC_f	Change in scale factor
ϕ_o	Orbital Phase
ϕ_a	Annual Phase
Φ_{in}	Flux Input
Φ_{max}	Maximum Input Flux
σ	Standard error in each measurment
$\sigma_{A,R,S,C}$	Standard error in A, R, S, and C
ω	Angular Frequency
ω_c	Angular calibration signal frequency
ω_r	Angular satellite roll frequency
A	Amplitude
A_c	Amplitude of calibration signal
A_o	Amplitude of orbital aberration
B_L	London moment magnetic field
B_{max}	Maximum trapped magnetic field for a given range
B_T	Trapped magnetic field
b	Voltage Bias
C	Amplitude of calibration signal at cosine of annual frequency

C_f	Scale Factor – flux in pickup loop to measured voltage output
EW	Gyroscope misalignment perpendicular to plane of orbit (East-West)
EW_0	Gyroscope misalignment perpendicular to plane of orbit (East-West) at time t_0
H_i	Measurement matrix
l_1,l_2,l_3,l_4	Components of annual aberration
N	Number of measurements
NS	Gyroscope misalignment in plane of orbit (North-South)
NS_0	Gyroscope misalignment in plane of orbit (North-South) at time t_0
P	Covariance matrix
RSS	root square sum
R	Rate of Change of Calibration Signal Amplitude
R_{EW}	Gyroscope drift rate perpendicular to plane of orbit (East-West)
R_{NS}	Gyroscope drift rate in plane of orbit (North-South)
S	Amplitude of calibration signal at sine of annual frequency
SQUID	Superconducting Quantum Interference Device
t	Time
t_a	Annual period
t_i	Time at i^{th} sample
t_0	Reference time
V_{out}	Output Voltage
x	states
z	Measured SRE signal
z_i	Sampled SRE Signal at time t_i

1. Introduction

Requirement 5 of the Gravity Probe B Twelve Fundamental Requirements (T002) specifies the requirements on the noise and nonlinearity of the gyroscope readout system.

> **Requirement 5, Gyroscope Readout**
> The SG readout system shall have a linearity consistent with an error of less than 0.3 marcsec/year. The gyroscope readout system single-sided noise spectral density shall be less than 190 marcsec/sqrt(Hz) x sqrt(roll period/180 sec) x (130 Hz/spin speed).

This paragraph contains two requirements. The first requirement states that the nonlinearity of the gyroscope readout system shall be consistent with an error of less than 0.3 marcsec/year The second requirement specifies the readout noise. One purpose of this document is to verify the requirement on the nonlinearity of the gyroscope readout.

A second purpose of this document is to justify the Gravity Probe B Program Control Board Change Request #559, which changed the requirements on the allowable harmonic distortion of in the System Design and Performance Requirements (T003). With this change these requirements read

> **3.2.1 Linearity**
> The readout system shall meet the following linearity requirements in the presence of trapped flux levels as specified in requirement 1.5:
>
> **3.2.1.1 High Frequency**
> In the frequency range 100-1000 Hz the harmonic distortion of each of harmonics 2-5 of sinusoids with amplitude corresponding to the trapped flux levels of section 1.5 shall be less than 1.5e-4.
>
> **3.2.1.2 Low Frequency**
> For frequencies less than 1 Hz, the harmonic distortion of each harmonic 2-5 of sinusoids with amplitude up to 80 arcsec London moment equivalent shall be less than 1.5e-4.

For both requirements 3.2.1.1. and 3.2.1.2, the Program Control Board Change Request changed the maximum harmonic distortion from 1.0e-4 to 1.5e-4. This document demonstrates that T002, Requirement 5, is still satisfied with the increase of the allowable harmonic distortion.

2. Measurements of Harmonic Distortion

The nonlinearity of any system may be measured by injecting a sinusoidal signal into the input and measuring the harmonic distortion of the output. The System Design and Performance Requirements (T003) specify the harmonic distortion of a sinusoidal signal. These measurements are summarized in the "SRE Special Linearity Test Analysis", [1]. A low frequency signal at the satellite roll rate and a high frequency signal at the gyroscope spin frequency were injected into the pickup loop circuit and the harmonic distortion of the output was measured. The worst-case harmonic distortion while operating the SQUID Readout Electronics (SRE) in Range 1 or Range 2 is shown in Table 1 below. During science data collection, the SRE electronics will be operated on one of these two ranges. In the special linearity test, there was some evidence of aliasing of high frequency signals which increased the measured harmonic distortion. The actual harmonic distortion is expected to be smaller than the values given here.

Table 1. Measured Worst Case Harmonic Distortion

	Worst Case Harmonic Distortion (PPM)			
	Harmonic 2	Harmonic 3	Harmonic 4	Harmonic 5
Low Frequency	135	102	104	79
High Frequency	116	91	44	39

3. Relationship of Nonlinearity to Harmonic Distortion

The nonlinearity of any system may be specified by a set of nonlinearity coefficients ε_i, which are defined for a given maximum input. If the input is Φ_{in} and the maximum input is Φ_{max}, the output will be given by

$$V_{out} = C_f \Phi_{in} \left[1 + \varepsilon_2 \frac{\Phi_{in}}{\Phi_{max}} + \varepsilon_3 \left(\frac{\Phi_{in}}{\Phi_{max}} \right)^2 + \ldots \right] + b \qquad (1)$$

where C_f and b are respectively the scale factor and the bias of the readout. If the input is sinusoidal,

$$\Phi_{in} = A\cos\omega t$$

with amplitude A and angular frequency ω, the output will be given by

$$V_{out} = C_f \left[\begin{array}{l} A\left(\frac{\varepsilon_2}{2}\frac{A}{\Phi_{max}} + \frac{3\varepsilon_4}{8}\frac{A^3}{\Phi_{max}^3} + \ldots \right) + A\cos\omega t \left(1 + \frac{3\varepsilon_3}{4}\frac{A^2}{\Phi_{max}^2} + \frac{5\varepsilon_5}{8}\frac{A^4}{\Phi_{max}^4} + \ldots \right) + \\ + A\cos 2\omega t \left(\frac{\varepsilon_2}{2}\frac{A}{\Phi_{max}} + \frac{\varepsilon_4}{2}\frac{A^3}{\Phi_{max}^3} + \ldots \right) + A\cos 3\omega t \left(\frac{\varepsilon_3}{4}\frac{A^2}{\Phi_{max}^2} + \frac{5\varepsilon_5}{16}\frac{A^4}{\Phi_{max}^4} + \ldots \right) + \\ + A\cos 4\omega t \left(\frac{\varepsilon_4}{8}\frac{A^3}{\Phi_{max}^3} + \ldots \right) + A\cos 5\omega t \left(\frac{\varepsilon_5}{16}\frac{A^4}{\Phi_{max}^4} + \ldots \right) + \ldots \end{array} \right] + b$$

From this equation, it can be seen that the measured harmonic distortion may be used to place limits on the nonlinearity coefficients. Since the amplitude of the input signal, A, is always less than Φ_{max}, the first term in the expression for the amplitude of each harmonic will dominate as long as the nonlinearity coefficients are approximately the same size. With this approximation, the ratio of the amplitude at the n^{th} harmonic to the fundamental will be given by

$$\frac{A(n\omega)}{A(\omega)} \approx \frac{\varepsilon_n}{2^{n-1}} \frac{A^{n-1}}{\Phi_{max}^{n-1}}$$

Then, the magnitude of any one of the nonlinearity coefficients may be estimated by

$$\varepsilon_n \approx 2^{n-1} \frac{\Phi_{max}^{n-1}}{A^{n-1}} \frac{A(n\omega)}{A(\omega)}$$

Using this equation, the worst-case nonlinearity coefficients may be determined from the measured harmonic distortion. For the high frequency signals, the sinusoidal signal was 90% of the maximum amplitude, while for the low frequency signal the maximum amplitude was equivalent to an angular misalignment of 80 arc seconds or approximately 80% of the full scale range. These worst-case nonlinearity coefficients are listed in Table 2. To verify requirement T002, #5, on the linearity of the gyroscope readout, it is necessary to show that this requirement is met, using the values of the nonlinearity coefficients given in Table 2.

Table 2. Nonlinearity Coefficients Derived from Measured Harmonic Distortion

	ε_2	ε_3	ε_4	ε_5
Low Frequency	3.4×10^{-4}	6.4×10^{-4}	1.6×10^{-3}	3.1×10^{-3}
High Frequency	2.6×10^{-4}	4.5×10^{-4}	4.8×10^{-4}	9.5×10^{-4}

For comparison the nonlinearity coefficients derived from the revised harmonic distortion requirements in System Design and Performance Requirements (T003) are listed in Table 3. Note that the difference between required values for the low frequency and high frequency distortion coefficients is because the amplitude of the injected signal is assumed to be 80% of full scale for the low frequency measurements and 90% of full scale for the high frequency measurements. To justify the change in T003 requirements 3.2.1.1 and 3.2.1.2, it is necessary to show that T002, Requirement 5 is met with the values of the nonlinearity coefficients given in Table 3.

Table 3. Nonlinearity Coefficients Derived from the Required Harmonic Distortion

	ε_2	ε_3	ε_4	ε_5
Low Frequency	3.75×10^{-4}	9.4×10^{-4}	2.3×10^{-3}	5.9×10^{-3}
High Frequency	3.3×10^{-4}	7.4×10^{-4}	1.6×10^{-3}	3.7×10^{-3}

4. Effects of Nonlinearity on the Science Gyroscope Readout

Nonlinearity of the gyroscope readout will produce two distinctly different effects on the output. First, it will mix the various input signals to produce output signals at frequencies which are the sums and differences of the input signals. Secondly, it will change the scale factor of the system. The effect of the nonlinearity of the gyroscope readout system on the signals at the harmonics of the gyroscope spin speed are discussed in reference [2], and the effects of the nonlinearity of the gyroscope readout system on the signals at the gyroscope roll rate are discussed in reference [3]. The effects discussed in each of these documents is summarized here.

4.1 Effects of Nonlinearity on the Signals at the Harmonics of the Gyroscope Spin Speed

4.1.1 Rectification of High Frequency Signals

In [2], the London moment equivalent angle of the d.c. rectified signal due to the signals at the harmonics of the gyroscope spin speed was shown to be

$$\beta_{dc} \leq \frac{\varepsilon_2}{2} \frac{B_T^{\,2}}{B_L B_{\max}}$$

where ε_2 is the quadratic high frequency nonlinearity coefficient given in Table 2, B_T is the trapped magnetic flux, B_L is the London moment field, and B_{max} is the trapped magnetic field which gives a full scale readout on a given range. Using the values B_T=8.3 μgauss (worst-case) [4], B_L=100 μgauss at 150 Hz spin speed [5], B_{max}=9 μgauss in range 2 (T003, Req. 1.5 and [6]), the London moment equivalent angle is 2.1 arc seconds. Since the magnitude of signals at the odd harmonics of the gyroscope spin speed will be modulated at the polhode frequency, this d.c.bias will also be modulated at the polhode frequency. Note that this effect has a similar time signature as the effect of a misalignment of the plane of the pickup loop from the satellite roll axis [5]. Both of these effects produce a modulation of the output of the gyroscope readout system at harmonics of the polhode frequency.

The satellite roll rate will be chosen to be incommensurate with the polhode frequency. If these signals at the polhode frequency are not included in the data analysis, there is the potential for leakage from the harmonics of polhode frequency into the measured amplitude of the roll frequency signal [7]. This leakage depends on the amplitude of the interfering signal, the frequency difference between the signal of interest and the interfering signal, and the length of the data set. For sufficiently small amplitudes at frequencies well separated in comparison to the inverse of the length of the data set, the leakage will be small enough that it can be neglected. If there are harmonics of the polhode frequency close to the satellite roll frequency, and they are included in the data analysis, their effects will be effectively removed. In either case, there will be no contribution to the roll frequency amplitude or phase from this modulation.

4.1.2 Modulation of the Low Frequency Scale Factor

Also in [2], the modulation of the magnetic flux applied to the SQUID at the harmonics of the gyroscope spin speed and the nonlinearity of the SQUID can change the low frequency scale factor of the gyroscope readout. This effect will have no impact on the overall experiment error because the magnitude of the fractional change in the scale factor is less than 10^{-7} and because the scale factor averaged over the polhode motion is expected to be constant.

4.2 Effects of Nonlinearity on Low Frequency Signals

4.2.1. Linearity of the Measurement Model

As shown in [3], the linear approximations made in the derivation of the measurement model introduce no additional errors greater than 10^{-13} radians in the bias. These errors are small enough to be neglected.

4.2.2 Changes in the Scale Factor That Depend on the Bias

The changes in the scale factor of the low frequency gyroscope readout may be classified as those that depend on the bias and those that depend on the amplitude of the signals. The magnitude of the scale factor changes that depend on the bias are shown in [3] to be

$$\frac{\delta C_f}{C_f} = \varepsilon_2 \frac{2\beta_b}{\beta_L^{\max}} + \varepsilon_3 \frac{3\beta_b^{\ 2}}{\left(\beta_L^{\max}\right)^2} + \ldots$$

where β_b is the London moment equivalent angle of any bias in the output of the SQUID readout electronics and β_L^{max} is the full scale London moment equivalent angle for a given range and gain of the SQUID readout electronics. An on-board Science Mode Offset Adjust Algorithm [8] keeps the bias within a specified fraction of the maximum value and thereby reduces these changes in the scale factor. If the Offset Adjustment Algorithm keeps the bias within 5% of the full scale range and the nonlinearity coefficients derived from the required values for the harmonic distortion are used, the maximum change in the scale factor is less than 4.4×10^{-5}. Using the nonlinearity coefficients derived from the measured harmonic distortion, the maximum fractional change in the calibration signal is 3.9×10^{-5}. The linear drift in the scale factor and the annual variation in the scale factor will be a small fraction of this maximum change.

In addition, since the scale factor change at the calibration signal frequency is the same as the scale factor change at the satellite roll rate, the calibration signal may be used to monitor the change in the scale factor at the satellite roll frequency. Appendix A is a covariance analysis of the accuracy with which a linear drift in the scale factor and an annual variation in the scale factor may be determined from measurements of the calibration signal over the course of one year. With a SQUID noise of 190 mas/√Hz and a calibration signal with a London moment equivalent amplitude of 30 arc seconds, the amplitude of the calibration signal may be determined to a fractional accuracy of 1.6×10^{-6}. A linear drift and a sinusoidal variation may also be estimated at the same time. The statistical error in the amplitude of each of these variations divided by the amplitude of the calibration signal is respectively 5.5×10^{-6} and 2.3×10^{-6}.

The effects of an unmodeled linear drift or annual variation in the scale factor have been calculated in [9]. An unmodeled fractional linear drift in the scale factor of 5.5×10^{-6} will produce an error in the drift rate measurement of less than 0.07 mas/yr. An unmodeled sinusoidal variation in the scale factor of 2.3×10^{-6} will produce a error in the measurement of the drift rate of less than 0.04 mas/yr. Both of these errors depend on the start date of the data acquisition, and in each case the worst-case start date was used. Since these errors may be correlated, the combined error is taken to be the sum of these contributions, or 0.11 mas/yr. This value is a worst-case error associated with the bias variation and the nonlinearity.

There are several potential methods for further reducing this error in the data analysis. Since the bias is well known at any given time, it would be possible to look for a correlation between the bias and the scale factor. To the extent that this relation between the bias and scale factor are well known, the effects of bias variation on the scale factor may be removed. Another possibility is to search for a linear or annual variation in the scale factor in the data. Since the scale factor is multiplicative, these temporal variations produce additional time signatures that will be observable in the data.

4.2.3 Changes in the Scale Factor That Depend on the Signal Amplitudes

In addition to the changes in the scale factor that are proportional to the bias, the cubic nonlinearity terms produce a change in the gyroscope readout scale factor which depends on the amplitude of the roll frequency and calibration frequency signals themselves. As shown in [3], the fractional change in the scale factor at the satellite roll frequency will be

$$\frac{\delta C_f(\omega_r)}{C_f} = \varepsilon_3\, \delta\left[\frac{3}{4}\left(\frac{\beta_L}{\beta_L^{\max}}\right)^2 + \frac{3}{2}\left(\frac{\beta_C}{\beta_L^{\max}}\right)^2\right] + \ldots$$

and the fractional change in the scale factor at the calibration signal frequency is given by

$$\frac{\delta C_f(\omega_C)}{C_f} = \varepsilon_3\, \delta\left[\frac{3}{2}\left(\frac{\beta_L}{\beta_L^{\max}}\right)^2 + \frac{3}{4}\left(\frac{\beta_C}{\beta_L^{\max}}\right)^2\right] + \ldots$$

where δ[ ] denotes the change in the quantity within the brackets. Since the change in the scale factor at the satellite roll frequency is not the same as the change in the scale factor at the calibration signal frequency, the amplitude of the calibration signal may not be used to monitor changes in the scale factor at the satellite roll frequency.

The magnitude of this contribution to the change in the scale factor may be estimated from the expected changes in the gyroscope spin axis. These expected changes include the effects of an average misalignment, NS_0 and EW_0, the linear drift rate of the gyroscope, R_{NS} and R_{EW}, and the orbital and annual aberration signals. The effects of the gravitational deflection of light and parallax are small enough that they may be neglected here. Then,

$$NS = NS_o + R_{NS}(t - t_0) + A_o \cos\phi_o + l_1 \cos\phi_a + l_2 \sin\phi_a$$
$$EW = EW_o + R_{EW}(t - t_o) + l_3 \cos\phi_a + l_4 \sin\phi_a$$

Here, A_o is the amplitude of the orbital aberration signal, and l_1, l_2, l_3, and l_4 are the components of the annual aberration signal. The phase of the orbital motion is denoted by ϕ_o and the annual motion by ϕ_a. When these terms are squared, only those changes in the scale factor at the orbital or annual frequencies or those that are linear in time will have a significant effect on the data analysis. The largest of these effects will be the term which arises from the average misalignment and the annual aberration. If the average misalignment is 1 arc sec (T002, Req. 6) and the amplitude of the annual aberration is 20 arc sec due to the motion of the earth about the sun, then the magnitude of the change in the scale factor at the annual frequency, will be given by

$$\left.\frac{\delta C_f(\omega_r)}{C_f}\right|_{annual} = \varepsilon_3 \frac{3}{4}\frac{2\langle\beta_L\rangle_{average}\langle\beta_L\rangle_{annual}}{\left(\beta_L^{\max}\right)^2} \approx 9.4\times10^{-4}\,\frac{3}{4}\frac{2\,(1\,\text{arc sec})(20\,\text{arc sec})}{(100\,\text{arc sec})^2} = 2.8\times10^{-6}$$

Using the results of [9], an unmodeled annual variation in the scale factor of this magnitude will give a worst case error in the gyroscope drift rate of 0.06 mas/yr. The value of nonlinearity coefficient used here is derived from the required value of the harmonic distortion. The measured values will further reduce this worst-case error.

As in the case of the scale factor variations that depend on the bias, there are additional possibilities for reducing these errors in the data analysis. The amplitude of the roll frequency signal will be well known at all times, and it will be possible to look for correlations between this amplitude and the scale factor. In addition, the data itself may be used to place limits on linear and annual scale factor variations.

Summary

The effects of nonlinearity on the gyroscope readout system may be summarized as follows:

1. High frequency nonlinearity of the gyroscope readout system at the odd harmonics of the gyroscope spin frequency will mix the large signals from the trapped magnetic flux to produced signals at d.c. and the even harmonics of the gyroscope spin speed. The worst case amplitude of these signals is expected to be on the order of 2.1 arc sec. Since these signals are incommensurate with the satellite roll rate, they are expected to have a negligible effect on the measurement of the gyroscope drift rate.

2. The flux applied to the SQUID at the odd harmonics of the gyroscope spin speed and the nonlinearity of the SQUID itself will change the open loop gain of the flux locked loop. The change in the scale factor of the gyroscope readout system associated with this change in the open loop gain is too small to have any significant impact on the experiment error. In addition, the value of this altered scale factor averaged over the polhode motion is expected to be stable.

3. The linear approximations used to derive the model for the gyroscope motion are sufficiently accurate that higher order terms do not need to be included.

4. The changes in the scale factor that depend on the bias may be monitored using the calibration signal. The accuracy with which the changes in the calibration signal may be determined in the presence of noise will determine the residual unmodeled changes in the scale factor. These unmodeled changes in the scale factor, in turn, contribute to the unmodeled error in the gyroscope drift rate. The RSS of the errors due to the residual uncertainty in the linear and annual variations of the scale factor is less than 0.08 mas/yr. Note that this error is applies to both the required and measured values of the nonlinearity coefficients.

5. Changes in the scale factor which are proportional to the amplitude of the roll frequency and calibration frequency signal amplitudes will produce errors in the drift rate of less than 0.06 mas/yr as long as the T003 requirements (1.5×10^{-4}) on the harmonic distortion are met.

The magnitude of these error estimates are worst-case values. In addition, the effects are uncorrelated, so it is appropriate to take the root square sum (RSS) of these errors. The RSS of the error in the gyroscope drift rate due to the nonlinearity determined from the required value of the harmonic distortion (1.5×10^{-4}) is 0.10 mas/year. The corresponding error due to the nonlinearity determined from the measured values of the harmonic distortion is slightly less than this value. Several approaches, which are described above, may be used to reduce this error in the ground-based data analysis with the flight data.

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Appendix A – Calibration Signal Covariance Analysis

The calibration signal is injected into the pickup loop circuit and its amplitude and phase, as measured by the SQUID Readout Electronics, may be used to monitor any changes in the scale factor and phase shift of the gyroscope readout system at the calibration signal frequency. The purpose of this covariance analysis is to calculate with what accuracy the amplitude, a linear drift in the amplitude, and an annual sinusoidal variation in the amplitude of the measured calibration signal frequency may be estimated. The accuracy of these components of the change in the calibration signal may be used to estimate the accuracy with which the changes in the scale factor may be estimated. The methods used here are similar to those used to calculate the analytical solution of the Gravity Probe B covariance matrix [10].

If there is a linear drift or an annual sinusoidal variation in the measured amplitude at the calibration signal frequency, then the measured calibration signal may be modeled as

$$z = \left(A_c + R\frac{t}{t_a} + S\sin\frac{2\pi t}{t_a} + C\cos\frac{2\pi t}{t_a} \right)\cos\omega_c t$$

Here t is time, t_a is one year, ω_c is the angular calibration signal frequency, and A_c, R, S, and C represent the amplitude, the rate of change of the amplitude, and variation in the amplitude and the sine and cosine of the annual frequency. The relation between any given measurement, z_i, these four states may be written as

$$z_i = H_i x$$

where H_i is the given by

$$H_i = \left[1 \;\; t_i/t_a \;\; \sin(2\pi t_i/t_a) \;\; \cos(2\pi t_i/t_a) \right]$$

and x is 4 × 1 matrix of the states of the system.

$$x^T = \left[A_c \; R \; S \; C \right]$$

The covariance matrix of the four states will then be given by

$$P = \sum_i \frac{H_i^T H_i}{\sigma^2}$$

where σ is the standard error in each measurement of z. This covariance matrix may be calculated by converting the sum to an integral and integrating over one year. For a one year period the off-diagonal terms are found to be zero. The diagonal terms give the standard error for each of the four states:

$$\sigma_{A_c}{}^2 = 2\sigma^2 / N$$

$$\sigma_R{}^2 = 24\sigma^2 / N$$

$$\sigma_S{}^2 = \sigma_C{}^2 = 4\sigma^2 / N$$

where N is the number of measurements of z.

The standard error of each of these parameters may be estimated for the data taken over one year assuming the noise in the gyroscope readout system only meets the required value of 190 mas/√Hz. If the measurements are made at one second intervals over the course of one year, σ=190 mas and N=3.15 × 10^7. The data for both guide star valid and guide star invalid may be used since the calibration signal may be used during both those periods. If the calibration signal has an amplitude equivalent to a London moment signal of 30 arc sec, then the corresponding fractional changes is the scale factor are

$$\left(\frac{\delta C_f}{C_f}\right)_R = \frac{\sigma_R}{A_c} = 5.5\times10^{-6}$$

$$\left(\frac{\delta C_f}{C_f}\right)_{S,C} = \frac{\sigma_{S,C}}{A_c} = 2.3\times10^{-6}$$

These values are used as the residual unmodeled linear and sinusoidal drift in the scale factor at an annual period.